On the high-energy behavior of nonlinear functionals of random eigenfunctions on $\mathbb{S}^d$

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19th European Young Statisticians Meeting
Prague, 31 Aug - 4 Sep 2015

outline

1. **Random eigenfunctions**
   - Why?
   - Our model: the d-dim sphere
2. **Nonlinear functionals of random eigenfunctions**
   - Our goal: high-energy behavior
   - Our tools: chaos expansion and 4th moment theorems
3. **Results**
   - Quantitative CLTs for Hermite rank 2-functionals
   - Application: Riemannian volume of excursion sets
4. **Overview: past and future**

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- $(\mathcal{M}, g)$= compact Riemannian manifold, $\Delta_{\mathcal{M}}$= Laplace-Beltrami operator
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- The excursion sets of f_j are defined, for $z \in \mathbb{R}$, as

$$\mathcal{A}_j(z) = \{x \in \mathcal{M} : f_j(x) > z\}.$$

Figure: **Ex excursion sets on the 2-sphere**

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- interesting topic

Ex : Yau's conjecture for the **nodal case** ($z = 0$)
$\exists$ constants $c_{\mathcal{M}}, C_{\mathcal{M}} > 0$ s.t.

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- Often, dealing with the **deterministic case is hard**!
  Indeed, great mathematicians worked on Yau's conjecture (Bruning-Gromes, Donnelly-Fefferman) but it is still open in its full generality.

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$$\mathrm{Cov}\,(W_j(x), W_j(y)) = J_0(\sqrt{E_j}|x-y|), \qquad x, y \in \mathbb{R}^2,$$

where J_0 is the Bessel function of order zero (Berry's Random Wave Model).

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- Conjecture: **The eigenfunction f_j of large eigenvalue $\approx E_j$ can be compared to the Gaussian field W_j related to $\sqrt{E_j}$.**

  Ex: For large j, the nodal set $f_j^{-1}(0)$ on the standard 2-torus, can be "compared" to the zero-set of the RWM W_j restricted to some suitable domain $U \subset \mathbb{R}^2$.

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- $\to$ increasing interest for special **random models** on 2-dim manifolds, e.g. the unit sphere $\mathbb{S}^2$

our model: random eigenfunctions on the d-sphere, $d \geq 2$

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- For each ℓ, **o.b. for the the ℓ-th eigenspace**: family of **hyperspherical harmonics** $\{Y_{\ell,m}, m = 1, \dots, n_{\ell;d}\}$, $n_{\ell;d} \approx \ell^{d-1}$
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- For $x \in \mathbb{S}^d$

$$T_\ell(x) := \sqrt{\frac{|\mathbb{S}^d|}{n_{\ell;d}}} \sum_{m=1}^{n_{\ell;d}} a_{\ell,m} Y_{\ell,m}(x)$$

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- $T_\ell = \{T_\ell(x)\}_{x \in \mathbb{S}^d}$ is the ℓ-th **random eigenfunction** on $\mathbb{S}^d$:

$$\Delta_{\mathbb{S}^d} T_\ell + \ell(\ell + d - 1) T_\ell = 0;$$

  isotropic, centered Gaussian field on $\mathbb{S}^d$ whose **covariance kernel** is

$$\mathrm{Cov}\,(\mathbf{T}_\ell(\mathbf{x}), \mathbf{T}_\ell(\mathbf{y})) = \mathbf{G}_{\ell;\mathbf{d}}(\cos \mathbf{d}(\mathbf{x}, \mathbf{y})), \qquad x, y \in \mathbb{S}^d, \qquad (1.1)$$

  where $d(x, y)$ is the distance between $x, y \in \mathbb{S}^d$ and $G_{\ell;d} = \alpha_{\ell,d} P_\ell^{(d/2-1, d/2-1)}$ is the **ℓ-th Gegenbauer polynomial** with $G_{\ell;d}(1) = 1$.

berry's rwm vs random spherical harmonics

- For $d = 2$ ($\mathbb{S}^2$), $G_{\ell;2} \equiv P_\ell$ the ℓ-th Legendre polynomial

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- By **Hilb's asymptotic formula**, as $\ell \to +\infty$,

$$\mathbf{P}_\ell(\cos d(x,y)) \approx \sqrt{\frac{d(x,y)}{\sin d(x,y)}}\,\mathbf{J_0}\big((\ell + 1/2)d(x,y)\big),$$

almost identical to RWM's covariance function, up to the factor

$$\sqrt{\frac{d(x,y)}{\sin d(x,y)}}$$

which seems to "remember" about the geometry of the 2-sphere.

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- High-energy behavior of T_ℓ (as $\ell \to +\infty$): **Geometry of excursion sets** $\mathcal{A}_\ell(z) = \{x \in \mathbb{S}^d : T_\ell(x) > z\}$ **for large** ℓ

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- In particular: Riemannian volume of $\mathcal{A}_\ell(z)$

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$$S_\ell(M) := \int_{\mathbb{S}^d} M(T_\ell(x))\, dx,$$

where $M : \mathbb{R} \to \mathbb{R}$ measurable s.t.

$$\mathbb{E}[M(Z)^2] < +\infty, \qquad Z \sim \mathcal{N}(0,1).$$

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- Note: $M(\cdot) = 1(\cdot > z) \quad \Rightarrow \quad S_\ell(M) =$ **Riemm-Vol**$(\mathcal{A}_\ell(z))$

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- H_q is the q-th Hermite polynomial, $S_\ell(M)[q] = \frac{J_q(M)}{q!} \int_{\mathbb{S}^d} H_q(T_\ell(x))\, dx$

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- IDEA: **1) asymptotic behavior (as $\ell \to +\infty$) of $\int_{\mathbb{S}^d} H_q(T_\ell(x))\, dx =: h_{\ell,q;d}$**
  2) deduce the asymptotic behavior of the whole series (2.1)

second tool: 4th moment theorems

CLT for $h_{\ell;q,d}$ with **rate of convergence**

- probability metrics between r.v.'s Z, N

$$Kolmogorov \quad d_K(Z,N) = \sup_{z\in\mathbb{R}} |\mathbb{P}(Z \le z) - \mathbb{P}(N \le z)| \ ,$$

$$TotalVariation \quad d_{TV}(Z,N) = \sup_{A\in\mathscr{B}(\mathbb{R})} |\mathbb{P}(Z \in A) - \mathbb{P}(N \in A)| \ ,$$

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- 4th moment theorem **Nourdin-Peccati,'12**: for $d_{\mathcal{D}} = d_{TV}, d_W, d_K$

$$d_{\mathcal{D}}\left(\frac{h_{\ell;q,d}}{\sqrt{\mathrm{Var}(h_{\ell;q,d})}}, Z\right) \le C_{\mathcal{D}}(q)\sqrt{\frac{\mathrm{cum}_4(h_{\ell;q,d})}{(\mathrm{Var}(h_{\ell;q,d}))^2}} \qquad (2.2)$$

where $Z \sim \mathcal{N}(0,1)$ and $C_{\mathcal{D}}(q) > 0$ some explicit constants

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  where $Z \sim \mathcal{N}(0,1)$ and $C_{\mathcal{D}}(q) > 0$ some explicit constants

- Therefore we have to study the variance of $h_{\ell;q,d}$ and the fourth cumulant of $h_{\ell;q,d}$ and show that the r.h.s. in (2.2) goes to 0, as $\ell \to +\infty$

asymptotic variance of $h_{\ell;q,d}$

- $\mathrm{Var}(h_{\ell;q,d}) = 2q!|\mathbb{S}^d||\mathbb{S}^{d-1}| \int_0^{\pi/2} G_{\ell;d}(\cos\vartheta)^q (\sin\vartheta)^{d-1}\, d\vartheta$.

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Proposition ($d=2$ Marinucci-Wigman,'11, $d\geq 3$ Marinucci-R.,'15)

As $\ell\to\infty$, for $d=2$ and $q=3$ or $q\geq 5$ and for $d,q\geq 3$,

$$\int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos\vartheta)^q(\sin\vartheta)^{d-1}\,d\vartheta = \frac{c_{q;d}}{\ell^d}(1+o(1)) \ .$$

The constants $c_{q;d}$ are given by the formula

$$c_{q;d} := \left(2^{\frac{d}{2}-1}\left(\frac{d}{2}-1\right)!\right)^q \int_0^{+\infty} J_{\frac{d}{2}-1}(\psi)^q \psi^{-q\left(\frac{d}{2}-1\right)+d-1} d\psi \ ,$$

where $J_{\frac{d}{2}-1}$ is the Bessel function of order $\frac{d}{2}-1$. Moreover for $c_{4;2}:=\frac{3}{2\pi^2}$

$$\int_0^{\frac{\pi}{2}} G_{\ell;2}(\cos\vartheta)^4 \sin\vartheta\, d\vartheta = \ c_{4;2}\frac{\log\ell}{\ell^2}(1+o(1)) \ .$$

$$\int_0^{\frac{\pi}{2}} G_{\ell;d}(\cos\vartheta)^2(\sin\vartheta)^{d-1}\,d\vartheta = 4\mu_d\mu_{d-1}\frac{c_{2;d}}{\ell^{d-1}}(1+o(1)) \ , \ c_{2;d} := \frac{(d-1)!\mu_d}{4\mu_{d-1}} \ .$$

quantitative clt's for $h_{\ell;q,d}$

Theorem (Marinucci-R.,2015)

For all $d, q \geq 2$ *and* $\mathcal{D} \in \{K, TV, W\}$ *we have, as* $\ell \to +\infty$,

$$d_{\mathcal{D}}\left(\frac{h_{\ell;q,d}}{\sqrt{\mathrm{Var}[h_{\ell;q,d}]}}, Z\right) = O\left(\ell^{-\delta(q;d)}(\log \ell)^{-\eta(q;d)}\right) \ , \tag{3.1}$$

where $Z \sim \mathcal{N}(0,1)$, $\delta(q;d) \in \mathbb{Q}$, $\eta(q;d) \in \{-1,0,1\}$ *and* $\eta(q;d) = 0$ *but for* $d = 2$ *and* $q = 4, 5, 6$.

- The exponents $\delta(q;d)$ and $\eta(q;d)$ can be given explicitly $\Rightarrow$ if $(d,q) \neq (3,3), (3,4), (4,3), (5,3)$ and $c_{q;d} > 0$,

$$\frac{h_{\ell;q,d}}{\sqrt{\mathrm{Var}(h_{\ell;q,d})}} \stackrel{\mathcal{L}}{\to} Z \ , \qquad \text{as } \ell \to +\infty \ , \tag{3.2}$$

  where $Z \sim \mathcal{N}(0,1)$.

For $d = 2$, the CLT (3.2) was already proved in Marinucci-Wigman'14; nevertheless (3.1) improves the existing bounds.

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- $\Rightarrow$ if $J_{\mathbf{2}}(M) = \mathbb{E}[M(Z)H_2(Z)] \neq 0$, then

$$\mathrm{Var}(S_\ell(M)) \sim \mathrm{Var}\left(\frac{J_2(M)}{2} h_{\ell;2,d}\right)$$

$S_\ell(M)$ and the summand $\frac{J_2(M)}{2} h_{\ell;2,d}$ have the same high-energy behaviour:

$$\frac{S_\ell(M) - \mathbb{E}[S_\ell(M)]}{\sqrt{\mathrm{Var}(S_\ell(M))}} = \frac{\frac{J_2(M)}{2} h_{\ell;2,d}}{\sqrt{\mathrm{Var}\left(\frac{J_2(M)}{2} h_{\ell;2,d}\right)}} + o_{\mathbb{P}}(1)$$

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- $\mathrm{Var}(h_{\ell;\mathbf{2},d}) \approx \ell^{-(\mathbf{d-1})}$, for $\mathbf{q \geq 3}$ $\mathrm{Var}(h_{\ell;\mathbf{q},d}) \approx \ell^{-\mathbf{d}}$
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$$\mathrm{Var}(S_\ell(M)) \sim \mathrm{Var}\left(\frac{J_2(M)}{2} h_{\ell;2,d}\right)$$

$S_\ell(M)$ and the summand $\frac{J_2(M)}{2} h_{\ell;2,d}$ have the same high-energy behaviour:

$$\frac{S_\ell(M)-\mathbb{E}[S_\ell(M)]}{\sqrt{\mathrm{Var}(S_\ell(M))}} = \frac{\frac{J_2(M)}{2} h_{\ell;2,d}}{\sqrt{\mathrm{Var}\left(\frac{J_2(M)}{2} h_{\ell;2,d}\right)}} + o_{\mathbb{P}}(1)$$

Theorem (Marinucci-R.'15)

$J_2(M) = \mathbb{E}\left[M(Z)H_2(Z)\right] \neq 0 \Rightarrow$ *quantitative CLT in Wasserstein distance*

$$d_W\left(\frac{S_\ell(M) - \mathbb{E}[S_\ell(M)]}{\sqrt{\mathrm{Var}(S_\ell(M))}}, Z\right) = O\left(\ell^{-\frac{1}{2}}\right), \qquad \textit{as } \ell \to \infty,$$

where $Z \sim \mathcal{N}(0,1)$.

application: riemannian volume

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Corollary (Marinucci-R.'15)

quantitative CLT in Wasserstein distance
As $\ell \to \infty$, *if* $z \neq 0$

$$d_W\left(\frac{\textit{Riemm-Vol}(A_\ell(z)) - \mu_d(1-\Phi(z))}{\sqrt{\mathrm{Var}(\textit{Riemm-Vol}(A_\ell(z)))}}, Z\right) = O\left(\ell^{-\frac{1}{2}}\right) .$$

In particular,

$$\frac{\textit{Riemm-Vol}(A_\ell(z)) - \mu_d(1-\Phi(z))}{\sqrt{\mathrm{Var}(\textit{Riemm-Vol}(A_\ell(z)))}} \xrightarrow{\mathcal{L}} Z$$

case $z = 0$: the defect

- Defect: difference between the measure of cold and hot regions

$$D_\ell := \int_{\mathbb{S}^d} 1(T_\ell(x) > 0)\, dx - \int_{\mathbb{S}^d} 1(T_\ell(x) < 0)\, dx \ ,$$

- $D_\ell = 2S_\ell(0) - |\mathbb{S}^d|$

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- R.'s PhD thesis (**Defect on $\mathbb{S}^d$, $d \ge 3$**)

$$\mathrm{Var}(D_\ell) = \frac{C_d}{\ell^{\mathbf{d}}}(1+o(1)) \ , \quad \text{as } \ell \to +\infty \ ,$$

  for $C_d > 0$. CLT: for $d \neq 3,4,5$, as $\ell \to +\infty$,

$$\frac{D_\ell}{\sqrt{\mathrm{Var}(D_\ell)}} \stackrel{\mathcal{L}}{\to} Z$$

- (1) **Riemannian volume on $\mathbb{S}^d$, $d \geq 2$** (Marinucci-Wigman,'11, Marinucci-Wigman,'14, Marinucci-R.'15, R.'15+)

overview on the geometry of high-energy excursion sets

- (1) **Riemannian volume on $\mathbb{S}^d$, $d \geq 2$** (Marinucci-Wigman,'11, Marinucci-Wigman,'14, Marinucci-R.'15, R.'15+)

- (2) Length of level curves on $\mathbb{S}^2$ (Marinucci-R.,'15+)
  Length of nodal lines on the 2-torus (Marinucci-Peccati-R.-Wigman,'15)

  (3) Euler-Poincaré characteristic on $\mathbb{S}^2$ (Cammarota-Marinucci-Wigman,'15)

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- FUTURE (?): geometry of eigenfunctions on any "nice" compact Riemannian manifold *thank you for your attention!*

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