Shape constrained inference

introduction & brief overview

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Outline

- Fitting a statistical model
- Isotonic regression
- Estimating a monotone density
- The current durations model
- Application to French fertility data
- More examples and directions

Fitting a statistical model

Observations: $X_1, \ldots, X_n$, i.i.d. $\sim F_0$
$F_0 \in \mathcal{F}$, statistical model

- $\mathcal{F}$: normal distribution functions
- $\mathcal{F}$: df's with $|F''| \leq 10$
- $\mathcal{F}$: concave df's on $[0, \infty)$
- $\mathcal{F}$: convex-concave df's on $[0, \infty)$

Observations: $(x_1, Y_1), \ldots, (x_n, Y_n)$, where

$$Y_i = r_0(x_i) + \epsilon_i$$

with $\epsilon_1, \ldots, \epsilon_n \sim N(0, \sigma^2)$, i.i.d. and $x_1 < \cdots < x_n$

Statistical model: $r_0 \in \mathcal{M}$

$\mathcal{M}$: $r_0(x) = \alpha + \beta x$

$\mathcal{M}$: $|r_0''|$ is continuous

$\mathcal{M}$: r_0 is increasing

Isotonic Regression

Data: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

$$x_1 < x_2 < \cdots < x_n$$

Least Squares estimate minimizes:

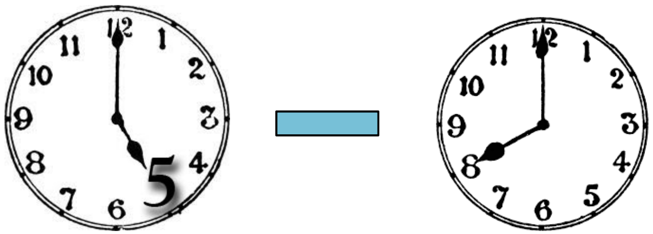
$$Q(r) = \frac{1}{2}\sum_{i=1}^{n}(r(x_i) - y_i)^2 \quad \text{such that } r \uparrow$$

Note: Q only depends on r *at* the x_i's

⇒ write r_i for $r(x_i)$

⇒ minimization over subset of $\mathbb{R}^n$

Examples



Solution to the minimization problem

$$Q(r) = \frac{1}{2}\sum_{i=1}^{n}(r_i - y_i)^2 \qquad r_1 \le r_2 \le \cdots \le r_n$$

$$\left.\begin{array}{l} \mathbf{1}.\ P_0 = (0,0) \\ \mathbf{2}.\ P_i = \left(i, \sum_{j=1}^{i} y_j\right),\ 1 \le i \le n \end{array}\right\} \quad \text{CSD}$$

3. Construct greatest convex minorant of

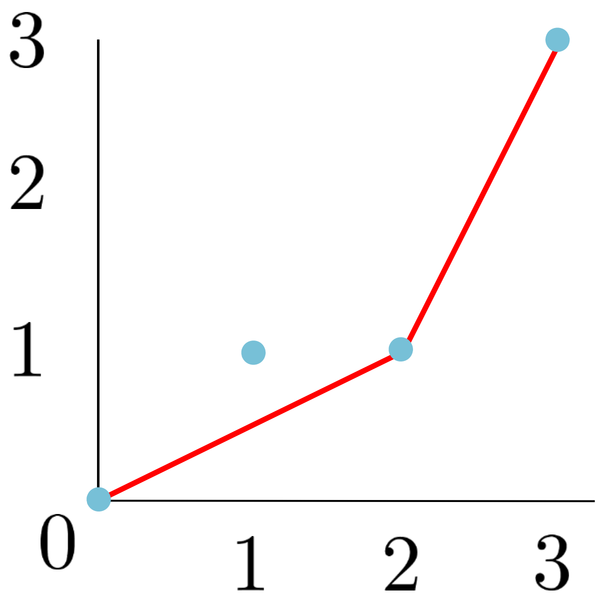
$$\{P_i \,:\, 0 \le i \le n\}$$

$\Rightarrow$ $\hat{r}_i$: left derivative of GCM at P_i

Example: $(2,1),(4,0),(7,2)$

1. $P_0 = (0,0)$ **2.** $P_i = \left(i, \sum_{j=1}^{i} y_j\right), \; 1 \le i \le n$

3. GCM of $\{P_i : 0 \le i \le n\}$



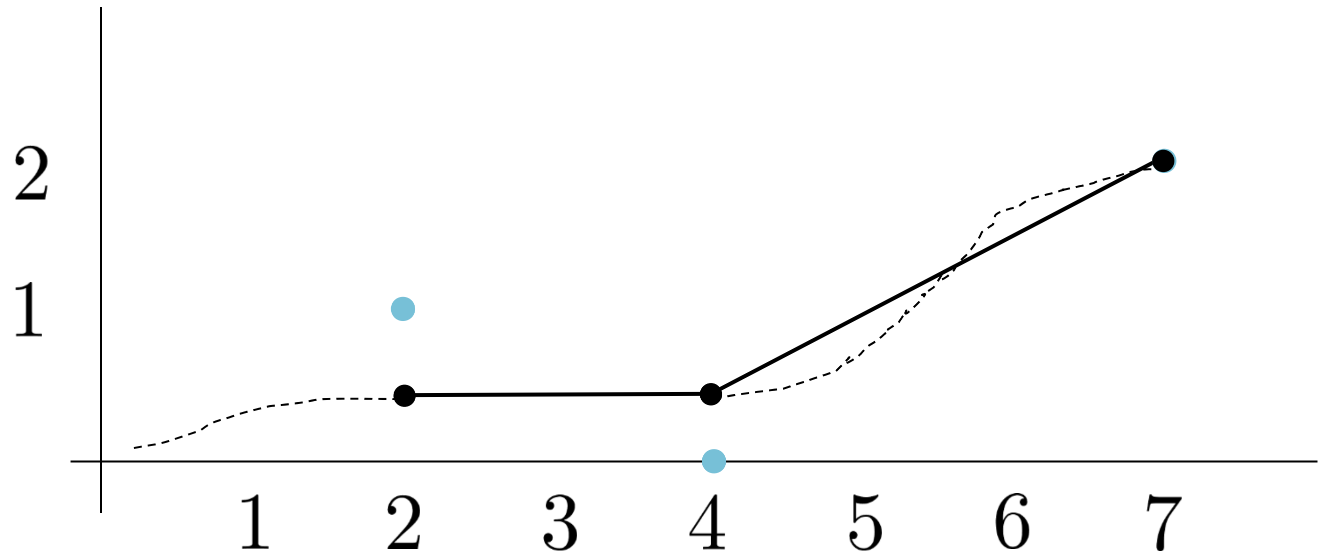
$$\hat{r}_1 = \frac{1}{2} = \hat{r}_2$$

$$\hat{r}_3 = 2$$

$\hat{r}_i$: left derivative of GCM at P_i

Example: $(2,1),(4,0),(7,2)$

$$\hat{r} = \left(\frac{1}{2}, \frac{1}{2}, 2\right)$$



Estimating a decreasing density

$X_1, X_2, \ldots, X_n \sim g$, i.i.d.

g: decreasing on $[0, \infty)$

Data: $x_1, x_2, \ldots, x_n$

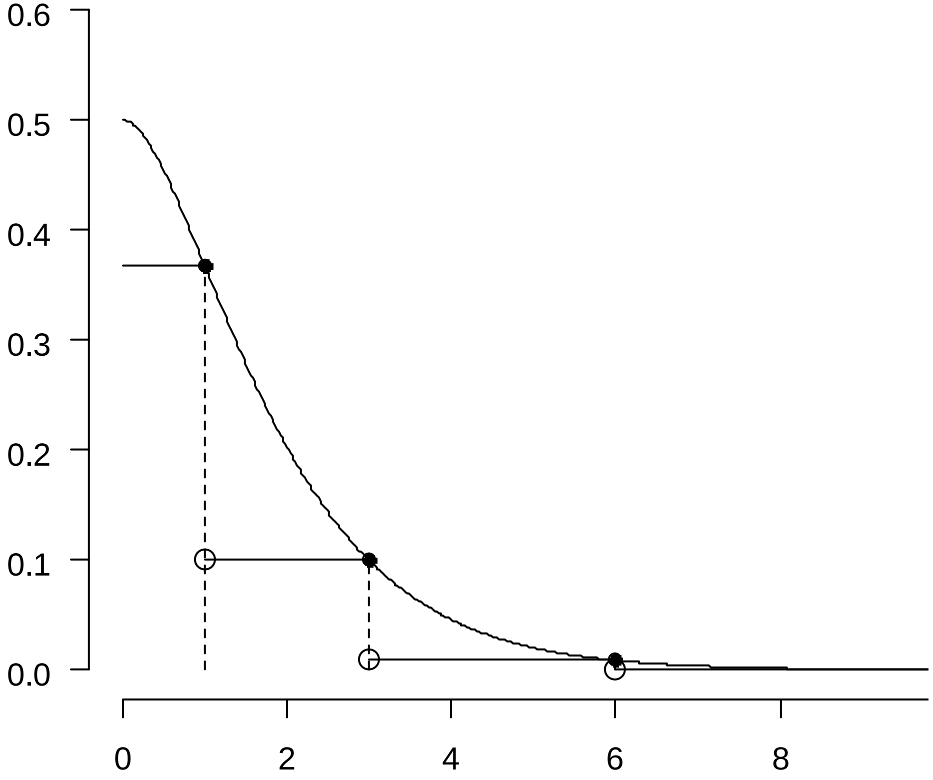
Estimate g based on $x_1, \ldots, x_n$

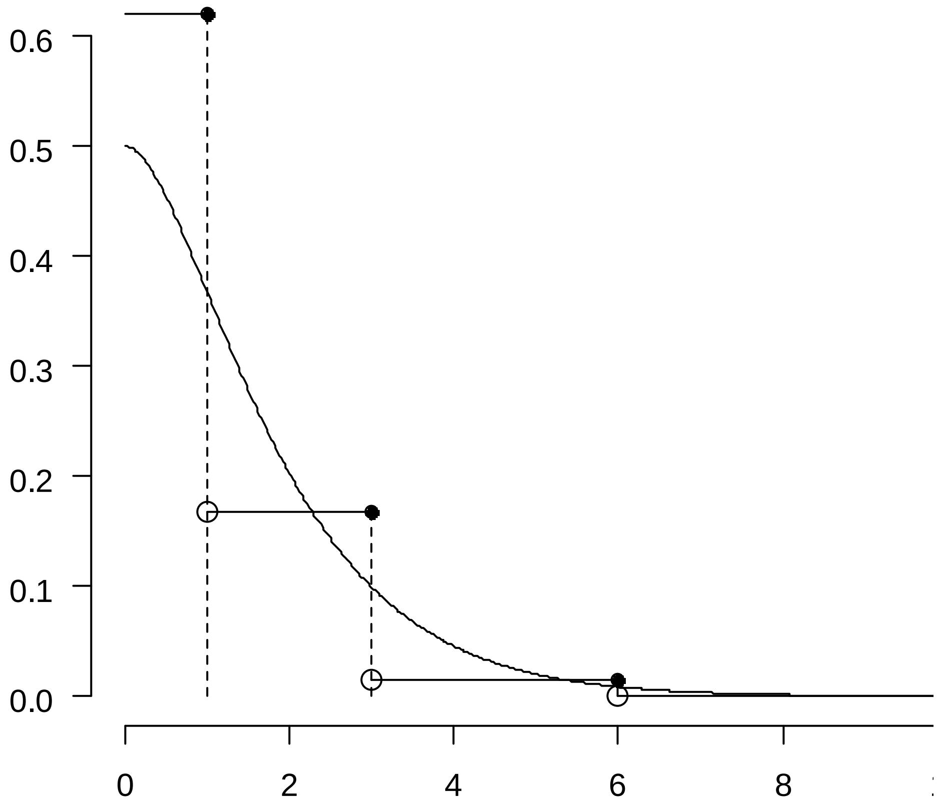
Log likelihood: $\ell(g) = \sum_{i=1}^{n} \log g(x_i)$

MLE for decreasing density

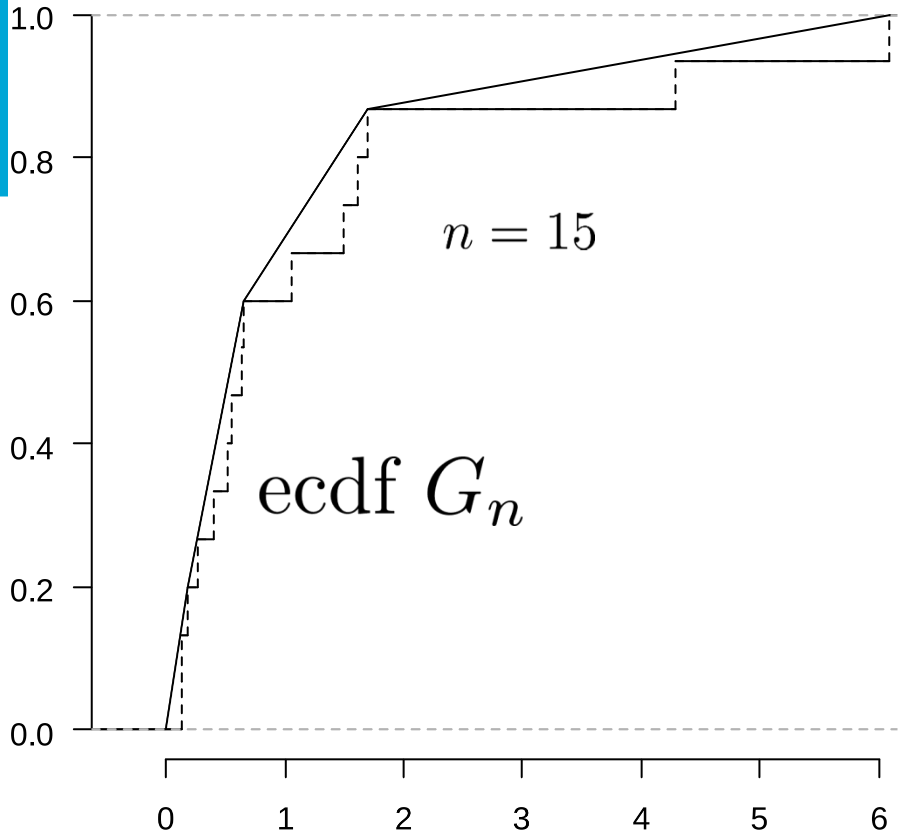
$$\ell(g) = \sum_{i=1}^{n} \log g(x_i)$$

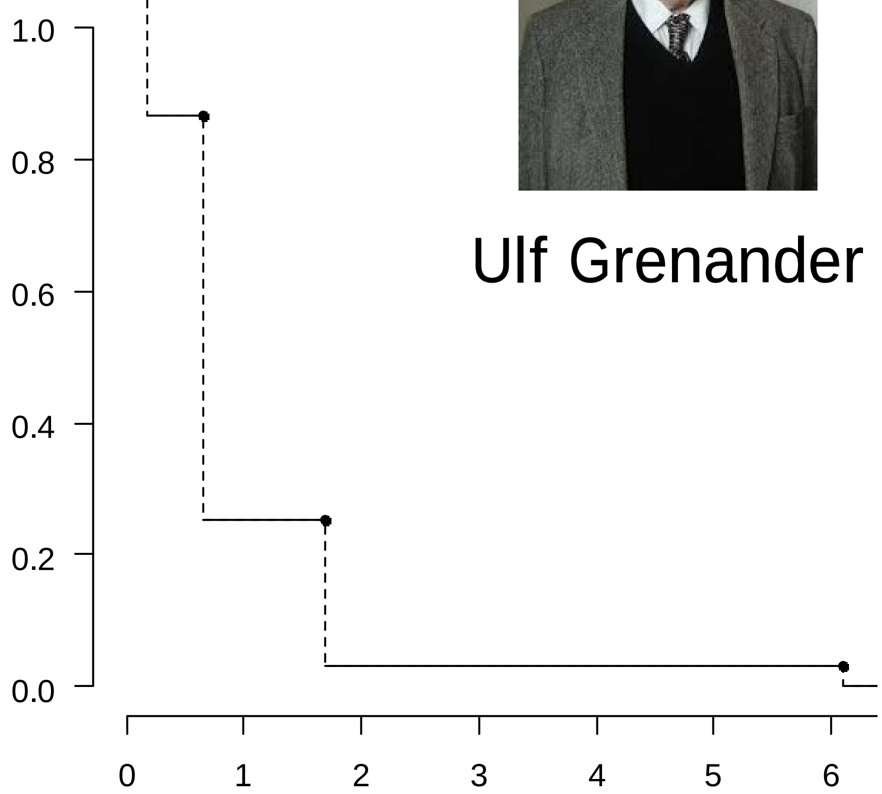
Maximizer: piecewise constant
left continuous





Construction of ML estimator





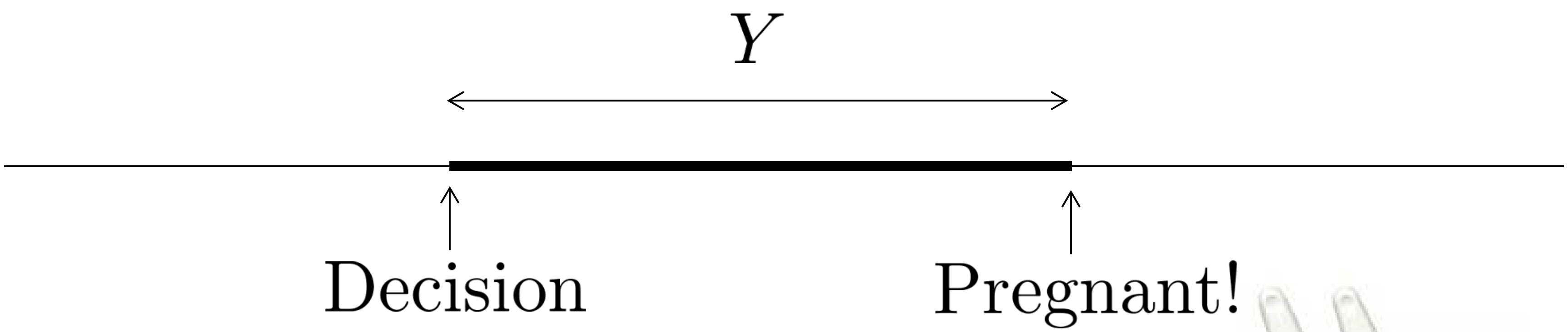
Ulf Grenander

Why would one be interested in

estimating a decreasing density?

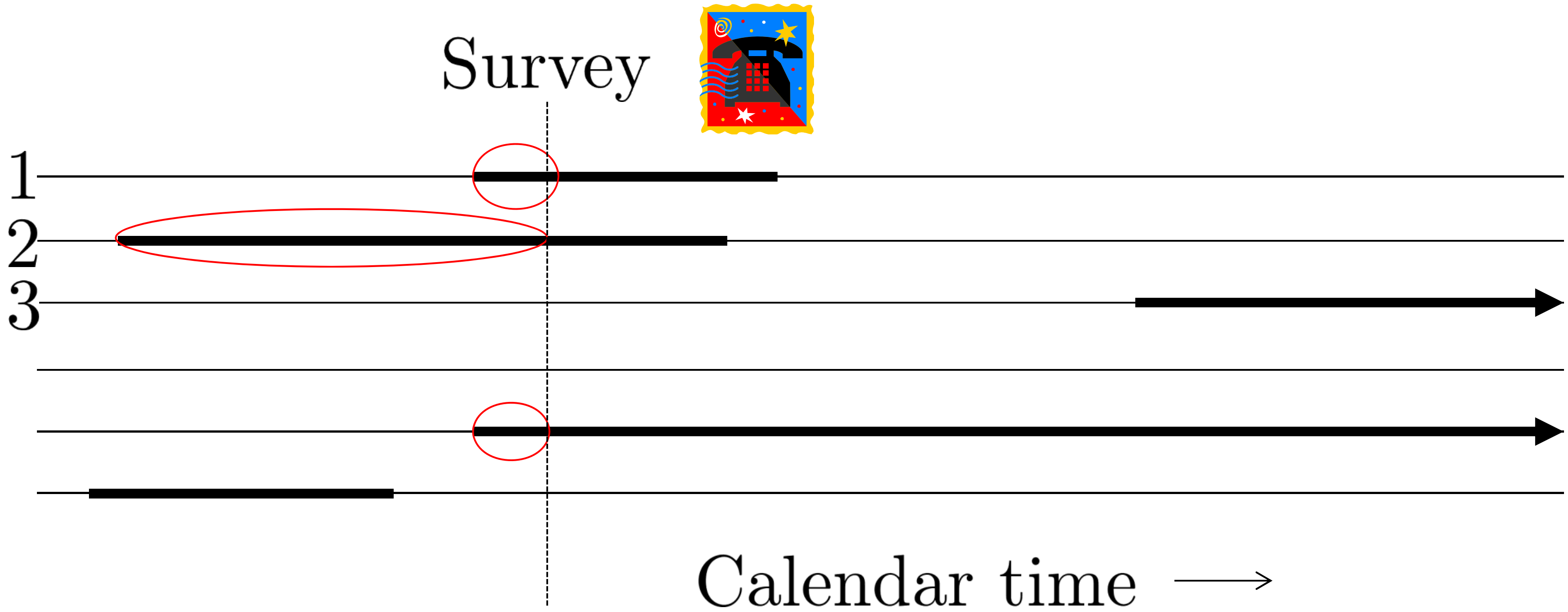
The current durations model

How long does it take to become pregnant?



Slama, R. *et al* (2012)
Estimation of the frequency of involuntary infertility on a nation-wide basis.
HUMAN REPRODUCTION

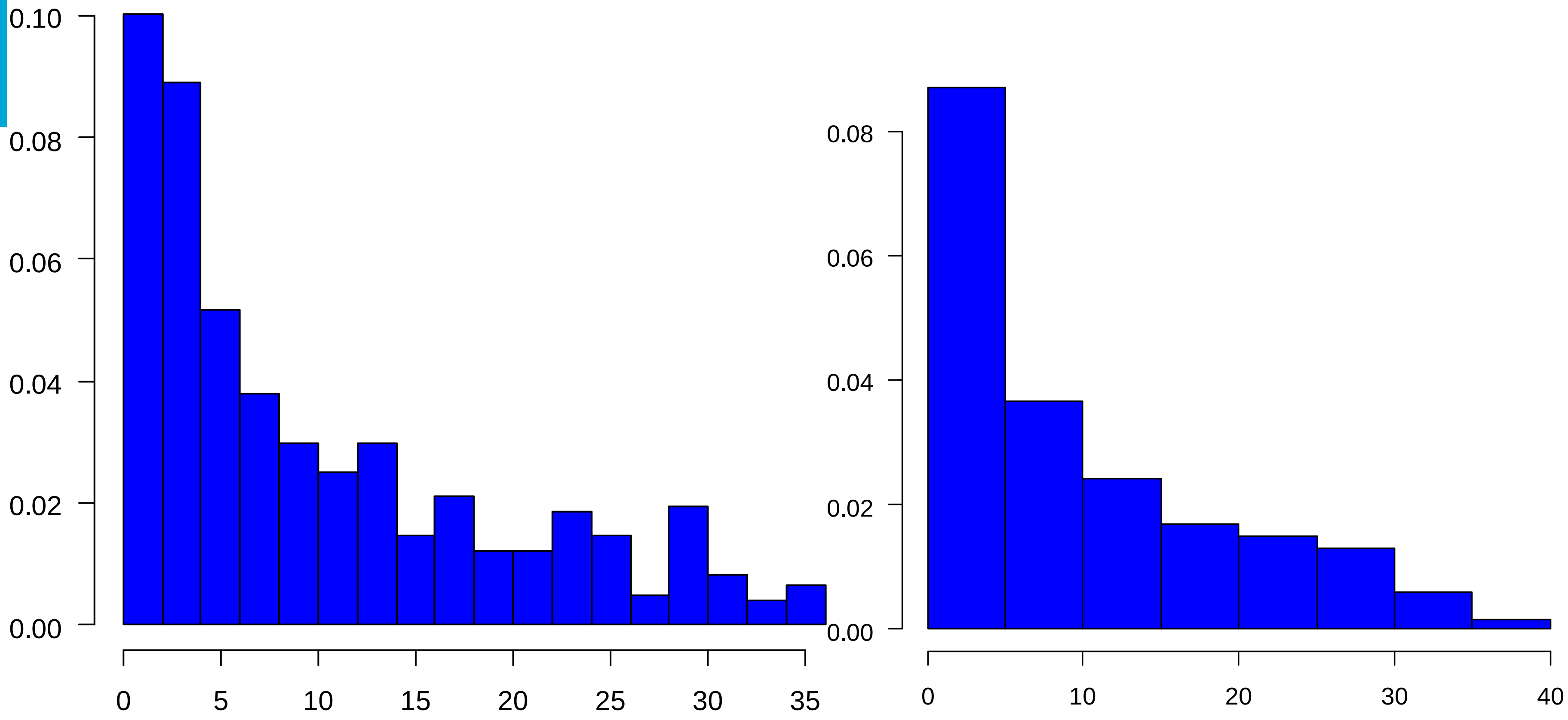
Cross-sectional study in France:



Question: when did 'your trial' start?

⟹ $X_1, X_2, \ldots, X_n$

Histogram of 618 values ≤ 36



Note: women with *larger* duration over-represented

Given the woman is in the sample, her current duration is draw from the *length biased* distribution associated with F_Y

$$\tilde{F}_Y(y) = \frac{\int_0^y x\, dF_Y(x)}{\int (1 - F_Y(x))\, dx}$$

X *smaller* than actual duration

It is a uniform random fraction of $\tilde{Y}$ taken from $\tilde{F}_Y$

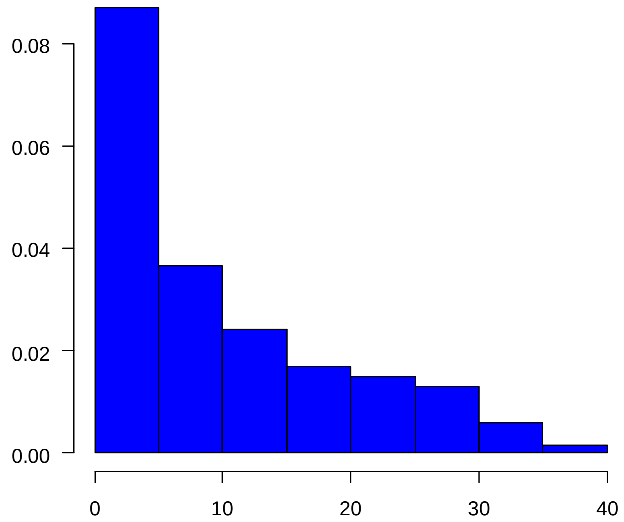
$$X = U\tilde{Y}$$

Relation distribution X and Y:

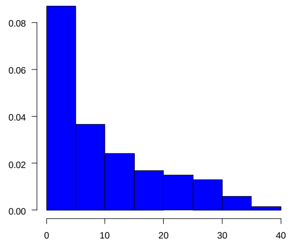
$$g_X(x) = \frac{1 - F_Y(x)}{\int y\, dF_Y(y)}, x \geq 0$$

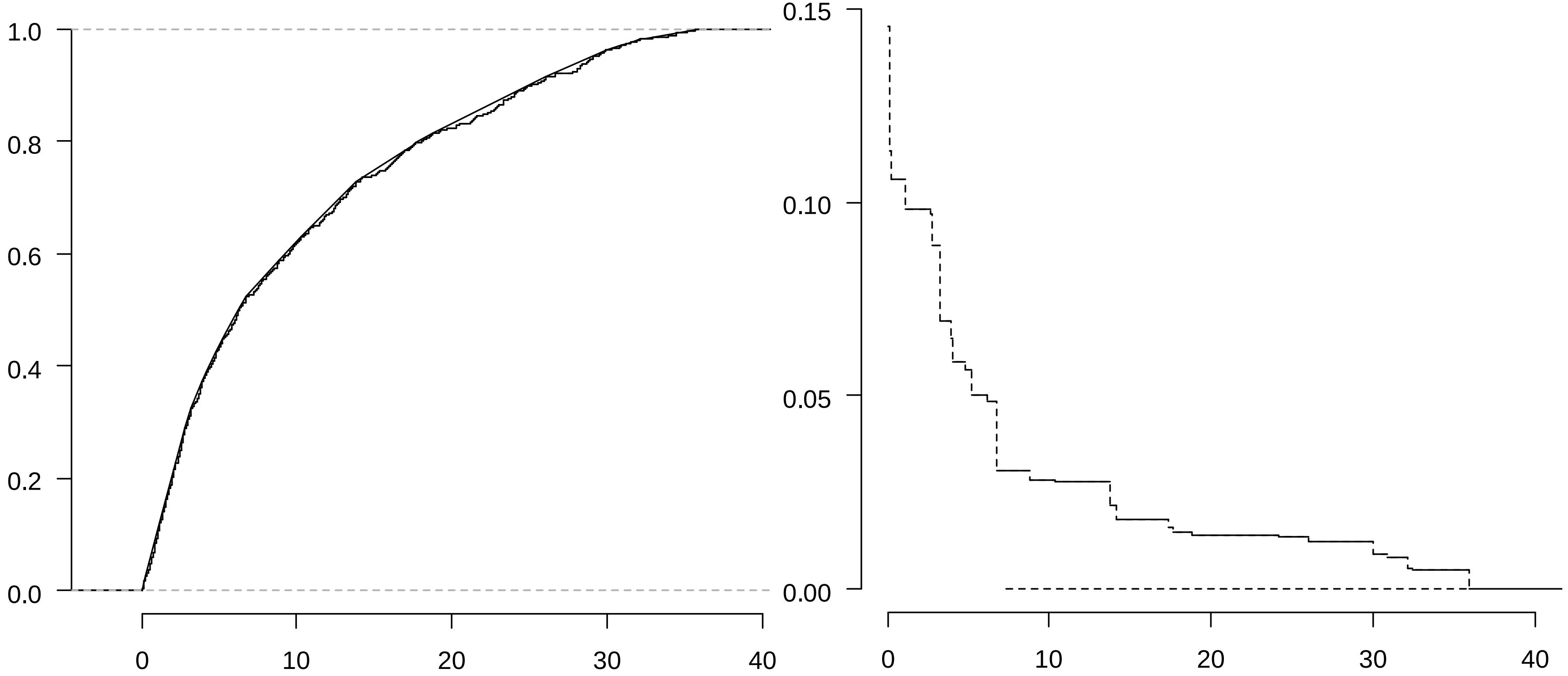
sampling density bounded, decreasing on $(0, \infty)$

$$F_Y(y) = 1 - \frac{g_X(y)}{g_X(0)}$$



MLE for the French fertility data

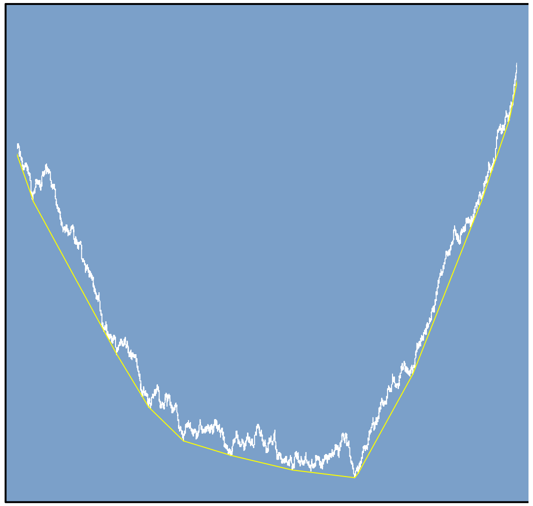


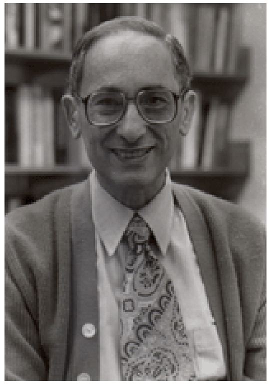


Asymptotic distribution of $\hat{g}_n(x)$ for $x > 0$.

$$n^{1/3} \frac{\hat{g}_n(x) - g_X(x)}{(g'_X(x) g_X(x)/2)^{1/3}}$$

$$\Rightarrow^D \text{ slocom } \left[t \mapsto W(t) + t^2\right]_{t=0}$$





Herman Chernoff

- Estimate $g_X(0)$
- Construct (pointwise) confidence band for g_X, F_Y

The MLE at zero

$$\hat{g}_n(0) = \max_{1 \le i \le n} \frac{\hat{G}_n(X_{(i:n)})}{X_{(i:n)}} = \max_{1 \le i \le n} \frac{i}{nX_{(i:n)}}$$

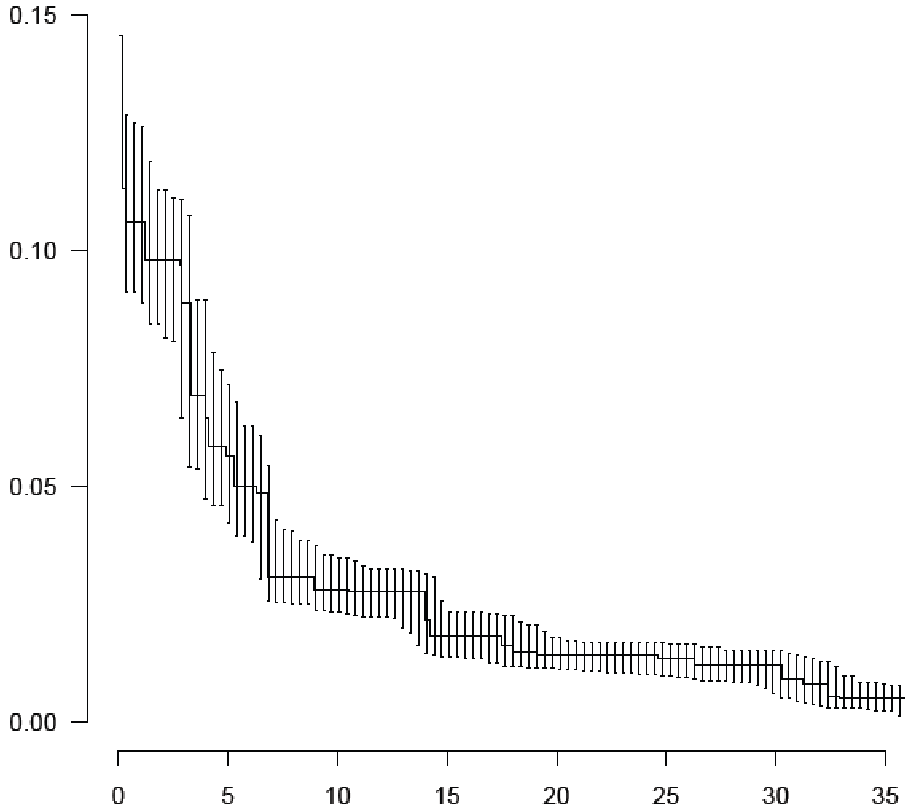
$$\ge \frac{1}{nX_{(1:n)}} = \frac{g_X(0)}{g_X(0)nX_{(1:n)}} \approx \frac{g_X(0)}{\mathrm{Exp}(1)}$$

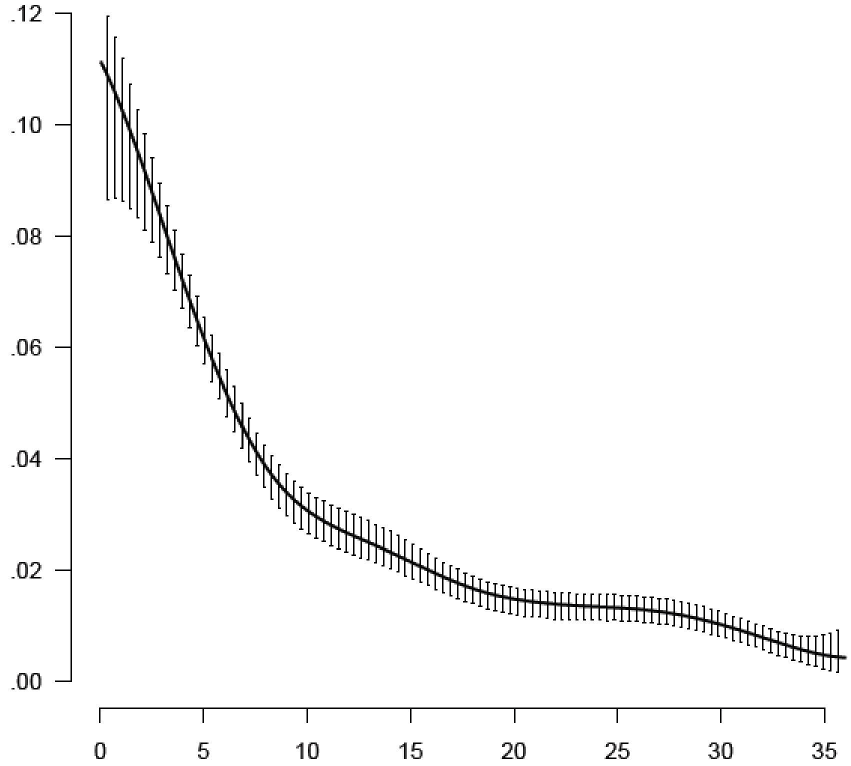
⇒ MLE is *inconsistent* at zero!

Confidence set for $g_X(x)$

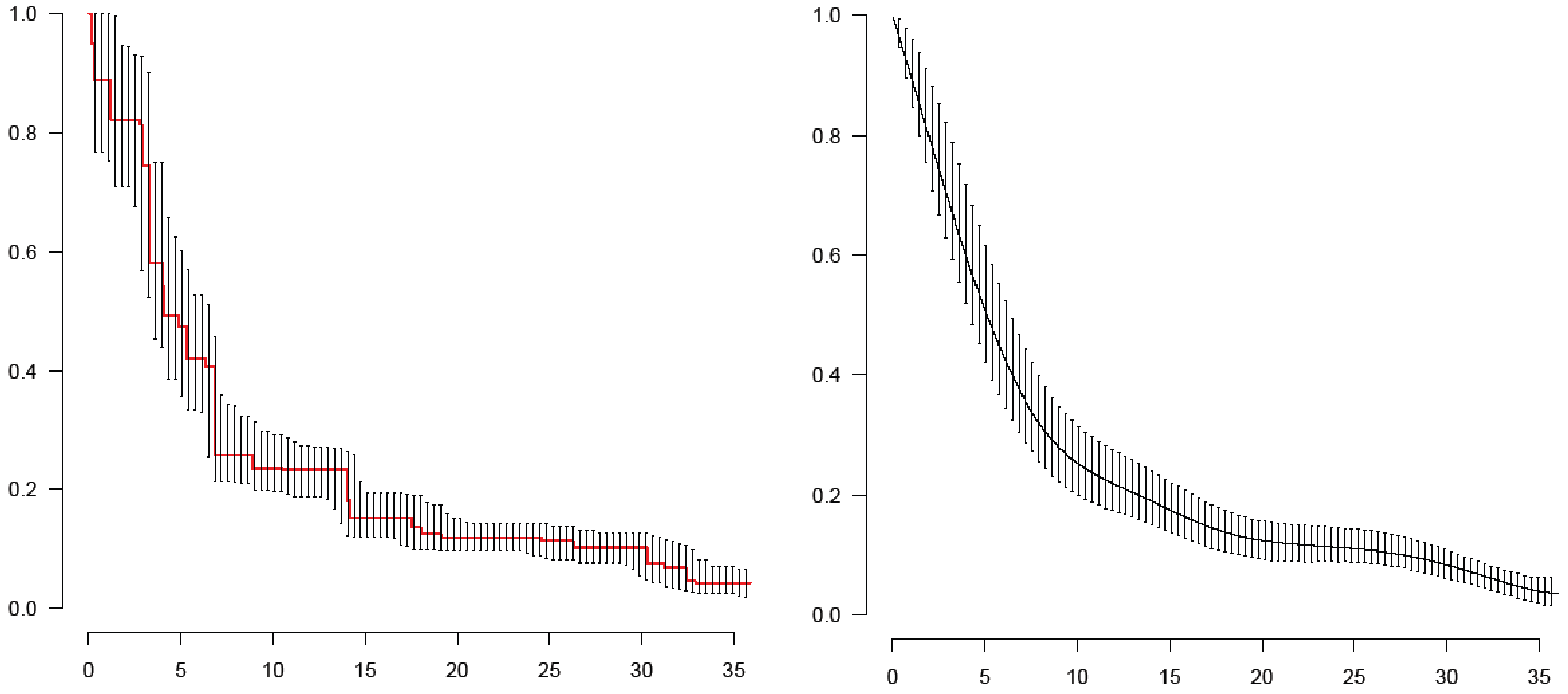
Likelihood Ratio approach

Bootstrap approach with smoothed estimate





Confidence sets for duration survival function $\frac{g_X(x)}{g_X(0)}$



More examples

The current status model

$$g(t,\delta) = \delta F(t)h(t) + (1-\delta)(1-F(t))h(t)$$

$$(t,\delta) \in [0,\infty) \times \{0,1\}$$

Min-uniform

$$G(z) = zF(z),\ z \in (0,1)\ \ F(x) = G(x)/x,\ x \in (0,1)$$

Competing risk with current status data

Bivariate interval censoring

Oriented cylinder model

many more

Summary

Small vs big statistical models

Isotonic regression: model and solution

Estimating a decreasing density: MLE

The current durations model

Some more examples

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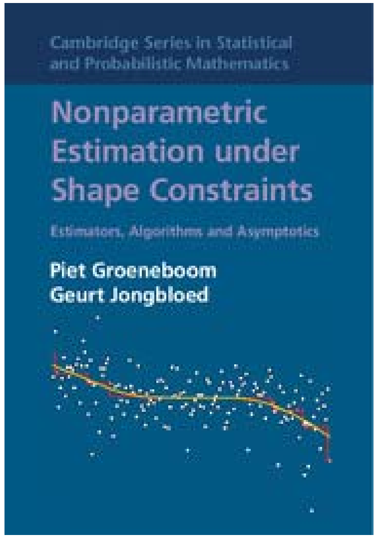

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Statistica Sinica

Thanks for your attention