The Weighted Log-Lindley distribution and its applications to lifetime data modeling

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19th European Young Statisticians Meeting

Prague, 31 August - 4 September 2015

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Lifetime data modeling

- Topic of real concern in many research fields: actuarial science, finance, medicine, engineering.
- Recently this issue received much attention from researchers and practitioners.
- The quality and effectiveness of the procedures used in a statistical analysis are determined by the assumed probability distribution.
- Recently, many lifetime distributions for modeling and analyzing data sets have been proposed.
- There still remain many important problems where the real data does not follow any of the existing probability distributions.
- Considerable effort has been expended in the development of large classes of new probability distributions along with relevant statistical methodologies.

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New results and main findings

- The Weighted Log-Lindley distribution, a new generalization of the Lindley distribution and an extension of the Log-Lindley distribution, generated by considering the weighting scheme.
- Mathematical properties
- Reliability analysis
- Statistical estimation
- Stochastic ordering
- Simulation algorithms
- The hazard rate function can have various shapes, providing a greater flexibility.
- The reliability behavior of the Weighted Log-Lindley distribution and its extensions allows a better performance for lifetime data modeling.

The Lindley distribution (L)

Definition

The r.v. X follows the Lindley distribution, $X \sim L(\sigma)$ if

$$F_X(x) = 1 - \left(1 + \frac{\sigma x}{\sigma + 1}\right) e^{-\sigma x},\ x > 0,\ \sigma > 0$$

$$f_X(x) = \frac{\sigma^2}{\sigma + 1}(1 + x)\, e^{-\sigma x},\ x > 0,\ \sigma > 0$$

Fact

The Lindley distribution $L(\sigma)$ is a mixture between $Exp(\sigma)$ and $\Gamma(2, \sigma)$, with mixing parameter $\frac{\sigma}{\sigma + 1}$.

Fact

The Lindley distribution provided the development of many applications to lifetime data modeling.

The hazard rate function (failure rate function)

Let X be a r.v. with p.d.f. f_X and survival function $S_X(x) = \mathrm{P}(X > x)$.

Definition

The **hazard rate function** (FR): $h_X(x) = \dfrac{f_X(x)}{S_X(x)}$, $x \in \mathbb{R}$ s.t. $S_X(x) > 0$.

- **Constant failure rate:** the probability of failure does not depend on the age; e.g.: the exponential distribution
- **Increasing failure rate:** the probability of failure increases with respect to the age; e.g.: most of the life and non-life systems
- **Decreasing failure rate:** the probability of failure decreases with respect to the age; e.g.: the lifetime of children in the first days of their lives

Fact

For lifetime modeling it is more plausible that FR behave different forms, depending on the parameters of the distribution.

The Lindley distribution: hazard rate function

Fact

The hazard rate function: $h_X(x) = \dfrac{\sigma^2(1+x)}{\sigma+1+\sigma x}$, $x > 0$, $\sigma > 0$

Advantages:

- Increasing hazard rate - more realistic than the Exponential distribution for lifetime data modeling
  $$\frac{\sigma^2}{\sigma+1} < h_X(x) < \sigma$$
- Better statistical properties.
- Shortcoming: does not provide enough flexibility for analyzing different types of lifetime data.

The Quasi Lindley distribution (QL)

Definition

The r.v. X follows the Quasi Lindley distribution, $X \sim QL(\sigma, \lambda)$ if

$$F_X(x) = 1 - \left(1 + \frac{\sigma x}{\lambda + 1}\right) e^{-\sigma x},\ x > 0,\ \sigma > 0,\ \lambda > 0$$

$$f_X(x) = \frac{\sigma}{\lambda + 1} (\lambda + \sigma x)\, e^{-\sigma x},\ x > 0,\ \sigma > 0,\ \lambda > -\sigma$$

Fact

The Quasi Lindley distribution $QL(\sigma, \lambda)$ is a mixture between $Exp(\sigma)$ and $\Gamma(2, \sigma)$, with mixing parameter $\dfrac{\lambda}{\lambda + 1}$.

The Quasi Lindley distribution: hazard rate function

Fact

The hazard rate function: $h_X(x) = \dfrac{\sigma(\lambda + \sigma x)}{\lambda + 1 + \sigma x}$, $x > 0$, $\sigma > 0$, $\lambda > 0$

- $\lambda = 0: \;\; QL(\sigma, 0) = L(\sigma)$
- Increasing hazard rate - more realistic for lifetime data modeling
  $\dfrac{\sigma\lambda}{\lambda + 1} < h_X(x) < \sigma$
- Better statistical properties

The Z-Generalized Lindley distribution (Z-GL)

Definition

The r.v. follows the Z-Generalized Lindley distribution, $X \sim QL(\sigma, \lambda, \alpha)$ if

$$f_X(x) = \frac{\sigma^2 (\alpha x)^{\alpha-1}}{(\lambda+\sigma)\Gamma(\alpha+1)} (\alpha + \lambda x)\, e^{-\sigma x},\ x > 0,\ \sigma > 0,\ \lambda > 0,\ \alpha > 0$$

Fact

$Z - GL(\sigma, \lambda, \alpha)$ is a mixture between $\Gamma(\alpha, \sigma)$ and $\Gamma(\alpha+1, \sigma)$, with mixing parameter $\frac{\lambda}{\lambda+\alpha}$.

Fact

$\alpha = 1 \Rightarrow Z - GL(\sigma, \lambda, 1) = QL(\alpha, \lambda)$

Weighted distributions

Fact

When data are recorded according to a certain stochastic model, the recorded observations will not have the original distribution unless every observation is given an equal chance of being recorded.

- Biased data arise in all domains of science. Often, sampling units cannot be selected with equal probability for statistical studies.
- The importance of using weighted distributions arises in such kind of situations.

Fact

Among the solutions for bias correction, weighted distribution theory gives a unified approach for modeling biased data.

The weighting scheme

Hypotheses:

- The original observation x_0 is modeled by a distribution with p.d.f. $f_0(x_0, \theta_1)$, with θ_1 parameter vector.
- Observation x is recorded according to a probability re-weighted by a weight function $w(x; \theta_2) > 0$, with θ_2 parameter vector.

The weighted model:

- Observation x is modeled using the distribution with p.d.f. $f(x) = Aw(x; \theta_2)f_0(x; \theta_1)$ - **weighted distribution**; A normalizing constant
- Patil and Rao, 1978 examined some general models leading to weighted distributions and showed how the weight $w(x; \theta_2) = x$ occurs in a natural way in many sampling problems.

Importance:

- provides a new understanding of standard distributions;
- provides methods of extending distributions for added flexibility in fitting data.

The Weighted Lindley distribution (WL)

Definition

$X \sim WL(\sigma, c)$ if

$$f_X(x) = \frac{\sigma^{c+1}}{(\sigma+c)\Gamma(c)} x^{c-1} (1+x) e^{-\sigma x}, x > 0, \sigma > 0, c > 0;$$

the weighting function is given by $w(x;c) = x^c$, $c > 0$.

Fact

1) $WL(\sigma, c)$ is a mixture between $\Gamma(c, \sigma)$ and $\Gamma(c+1, \sigma)$, with mixing parameter $\frac{\sigma}{\sigma+c}$.

2) The Weighted Lindley distribution provides more flexibility to Lindley distribution, in terms of the shapes of the p.d.f., hazard rate and mean residual life functions.

3) For some non-grouped/grouped survival data, the Weighted Lindley model is more suitable than some well known two-parameter survival models.

Weighted Lindley distribution - structural properties

Theorem

The p.d.f. of the Weighted Lindley distribution is:
1) decreasing, if $\{\sigma < c < 1;\ (\sigma + c)^2 - 4\sigma \leq 0\}$ *or* $\{c \leq 1, \sigma \geq c\}$;
2) unimodal, if $\{c = 1;\ \sigma < 1\}$ *or* $\{c > 1, \sigma > 0\}$;
3) decreasing-increasing-decreasing, if $\{\sigma < c < 1;\ (\sigma + c)^2 - 4\sigma > 0\}$.

Fact

Most classical two-parameter distributions, such as Weibull, Gamma and Gompertz distributions have either decreasing or unimodal densities. The p.d.f. of the weighted Lindley model adds an extra shape, which can be useful for modeling bimodal data.

Weighted Lindley distribution - hazard rate function

Lemma

Let X be a non-negative continuous r.v., with twice differentiable p.d.f. f and hazard rate function h. Let $\eta(x) = -\frac{d}{dx} \ln f(x)$.

1) If η is decreasing (increasing) in x, then h is decreasing (increasing) in x.

2) If η has a bathub (upside-down bathub) shape, then h has also a bathub (upside-down bathub) shape.

Theorem

The hazard rate function h of the WL distribution is bathub shaped if $0 < c < 1$ and increasing if $c \geq 0$, for all $\sigma > 0$.

Weighted Lindley distribution - mean residual life function

Lemma

Let X be a non-negative continuous r.v., with twice differentiable p.d.f. f, hazard rate function h and mean residual life function μ. If h is decreasing (increasing) in x, then μ is decreasing (increasing) in x.

Lemma

Let X be a non-negative continuous r.v., with twice differentiable p.d.f. f, hazard rate function h and mean residual life function μ. If h has a bathub (upside-down bathub) shape, then μ has also a bathub (upside-down bathub) shape.

Theorem

The mean residual life function μ of the WL distribution is upside-down bathub shaped if $0 < c < 1$ and increasing if $c \geq 0$, for all $\sigma > 0$.

The Log-Lindley distribution (L-L)

Definition

$X \sim L-L(\sigma, \lambda)$ if

$$f(x|\sigma,\lambda) = \frac{\sigma^2}{1+\lambda\sigma}(\lambda - \alpha \ln x)\, x^{\alpha-1};\ 0 < x < 1, \sigma > 0, \lambda \geq 0.$$

Advantages:

- The L-L distribution has a bounded domain and does not include any special function in its formulation, which enables the development of useful applications in actuarial science or econometric analysis.
- The L-L distribution provides a new alternative to the classical beta distribution, in addition to the existing ones G3B and G4B Generalized Beta distributions, which present the drawbacks of having too many parameters and involving special functions in their formulation.
- The L-L distribution induces a principle whose premium is between the net premium and the dual-power premium principles.

Stochastic ordering

- Many parametric families of distributions can be ordered by some stochastic orders according to the value of their parameters.
- The Log–Lindley distribution can be ordered in terms of **the likelihood ratio order**, which is a powerful tool in parametric models.

Definition

(Ross, 1996) Let X_1 and X_2 be continuous random variables with p.d.f. f_1 and f_2, respectively, such that $\frac{f_2}{f_1}$ is non-decreasing over the union of the supports of X_1 and X_2. Then X_1 is said to be smaller than X_2 in the likelihood ratio order (denoted by $X_1 \leq_{LR} X_2$).

Examples

1) Exponential: If $\gamma_1 < \gamma_2$, then $\mathrm{Exp}(\gamma_1) \leq_{LR} \mathrm{Exp}(\gamma_2)$.
2) Normal: If $\mu_1 \leq \mu_2$ and $\sigma_1 = \sigma_2$, then $\mathrm{N}(\mu_1, \sigma_1) \leq_{LR} \mathrm{N}(\mu_2, \sigma_2)$.
3) Binomial: If $n_1 \leq n_2$ and $p_1 \leq p_2$, then $\mathrm{Bi}(n_1, p_1) \leq_{LR} \mathrm{Bi}(n_2, p_2)$.

Stochastic ordering

Definition

Let X_1 and X_2 be random variables with c.d.f. F_1 and F_2, respectively. Then X_1 is said to be stochastically smaller than X_2 if $F_1(x) \geq F_2(x)$ for all x (denoted by $X_1 \leq_{ST} X_2$).

Definition

Let X_1 and X_2 be random variables with hazard rate functions r_1 and r_2, respectively. Then X_1 is said to be smaller than X_2 in the hazard order if $r_1(x) \leq r_2(x)$ for all x (denoted by $X_1 \leq_{HR} X_2$).

Fact

1) The likelihood ratio order is stronger than the hazard rate order.
2) The likelihood ratio order is stronger than the usual stochastic order.

Log-Lindley distribution: stochastic ordering

Theorem

Let X_1 and X_2 be Log–Lindley random variables with p.d.f. $f(x|\sigma_1, \lambda_1)$ and $f(x|\sigma_2, \lambda_2)$, respectively. If $\sigma_1 \leq \sigma_2$ and $\lambda_1 \leq \lambda_2$, then $X_1 \leq_{LR} X_2$.

Corollary

Let X_1 and X_2 be Log–Lindley r.v., with p.d.f. $f(x|\sigma_1, \lambda_1)$ and $f(x|\sigma_2, \lambda_2)$ and hazard rates r_1 and r_2, respectively. If $\sigma_1 \leq \sigma_2$ and $\lambda_1 \leq \lambda_2$, then:
1) $E\left[X_1^k\right] \leq E\left[X_2^k\right]$, for all $k > 0$.
2) $r_1(x) \leq r_2(x)$ for all $x \in (0, 1)$.

Interpretation: The mean of the Log–Lindley parametric family increases with the values of the parameters:

- Fixed λ, the mean increases as σ increases.
- Fixed σ, the mean increases as λ increases.

Log-Lindley distribution: stochastic ordering

Theorem

Let Y_1 and Y_2 be two binomial L–L r.v. with c.d.f. $G(x|\sigma_1, \lambda_1)$ and $G(x|\sigma_2, \lambda_2)$. If $\sigma_1 \leq \sigma_2$ and $\lambda_1 \leq \lambda_2$, then $Y_1 \leq_{LR} Y_2$.

Corollary

Let Y_1 and Y_2 be two binomial L–L r.v., with c.d.f. $G(x|\sigma_1, \lambda_1)$ and $G(x|\sigma_2, \lambda_2)$ and hazard rates r_1 and r_2. If $\sigma_1 \leq \sigma_2$ and $\lambda_1 \leq \lambda_2$, then:
1) $E\left[Y_1^k\right] \leq E\left[Y_2^k\right]$, for all $k > 0$.
2) $r_1(x) \leq r_2(x)$ for all $x \in (0, 1)$.

Importance:

- The L–L distribution can also be ordered by some stochastic orders depending on the values of their parameter.
- The L–L distribution can be used as mixing distribution in order to to derive univariate discrete distributions, with applications in different settings, by compounding the binomial distribution with L–L.

Log-Lindley distribution: stochastic ordering

Theorem

Let Z_1 and Z_2 be negative binomial L–L r.v. with c.d.f. $G(x|\sigma_1, \lambda_1)$ and $G(x|\sigma_2, \lambda_2)$. If $\sigma_1 \leq \sigma_2$ and $\lambda_1 \leq \lambda_2$, then $Z_1 \leq_{LR} Z_2$.

Corollary

Let Z_1 and Z_2 be negative binomial L–L r.v., with c.d.f. $G(x|\sigma_1, \lambda_1)$ and $G(x|\sigma_2, \lambda_2)$ and hazard rates r_1 and r_2. If $\sigma_1 \leq \sigma_2$ and $\lambda_1 \leq \lambda_2$, then:
1) $E\left[Z_1^k\right] \leq E\left[Z_2^k\right]$, for all $k > 0$.
2) $r_1(x) \leq r_2(x)$ for all $x \in (0, 1)$.

Importance:

- The L–L distribution can be used as mixing distribution in order to to derive univariate discrete distributions, with applications in different settings, by compounding the negative binomial distribution with the L–L distribution.

The Weighted Log-Lindley distribution (WLL)

We propose a new extension of the L-L distribution, by considering the weighting function $w(x;c) = x^{c-1}$, $c > 0$.

Definition

$X \sim WLL(\sigma, \lambda, c)$ if $f(x|\sigma, \lambda, c) = \dfrac{(\sigma + c)^2}{1 + \lambda(\sigma + c)} (\lambda - \ln x)\, x^{\sigma + c - 1}$;
$0 < x < 1, \sigma > 0, \lambda \geq 0, c > 0.$

Fact

1) WLL provides more flexibility than L-L, in terms of the shapes of the p.d.f., hazard rate and mean residual life functions.
2) For some non-grouped/grouped survival data, the WLL model is more suitable than some well known three-parameter survival models.

The Weighted Log-Lindley distribution (WLL)

We consider the probability density function of the WLL distribution, given by: $f(x|\sigma, \lambda, c) = \dfrac{(\sigma + c)^2}{1 + \lambda(\sigma + c)} (\lambda - \ln x)\, x^{\sigma + c - 1}$;
$0 < x < 1, \sigma > 0, \lambda \geq 0, c > 0.$

Definition

The cumulative distributin function of the WLL distribution is given by:
$F(x|\sigma, \lambda, c) = \dfrac{x^{\sigma + c}[1 + (\sigma + c)(\lambda - \ln x)]}{1 + \lambda(\sigma + c)}$;
$0 < x < 1, \sigma > 0, \lambda \geq 0, c > 0.$

Weighted Log-Lindley distribution: the moments

Theorem

The moments and the inverse moments are given by:

$$E(X^r|\sigma,\lambda,c) = \frac{(\sigma+c)^2}{1+\lambda(\sigma+c)} \cdot \frac{1+\lambda(\sigma+r+c)}{(\sigma+r+c)^2}; \; r = \ldots, -2, -1, 1, 2, \ldots.$$

Corollary

The mean and the variance are given by:

$$E(X|\sigma,\lambda,c) = \frac{(\sigma+c)^2}{1+\lambda(\sigma+c)} \cdot \frac{1+\lambda(\sigma+c+1)}{(\sigma+c+1)^2}; \; \sigma>0, \lambda \geq 0, c>0;$$

$$Var(X|\sigma,\lambda,c) = \frac{(\sigma+c)^2}{1+\lambda(\sigma+c)} \left[\frac{\lambda(\sigma+c+2)+1}{(\sigma+c+2)^2} - \frac{\sigma^2\left[\lambda(\sigma+c+1)+1\right]^2}{[1+\lambda(\sigma+c)](\sigma+c+1)^4} \right].$$

Weighted Log-Lindley distribution - structural shape properties

$$f(x|\sigma,\lambda,c)=\frac{(\sigma+c)^2}{1+\lambda(\sigma+c)}(\lambda-\ln x)\,x^{\sigma+c-1};$$
$$0<x<1,\sigma>0,\lambda\geq 0,c>0$$

Fact

For $\lambda \geq 0$ and $0 \leq \sigma + c \leq 1$, f has U-shape and it is asymptotic to 0.

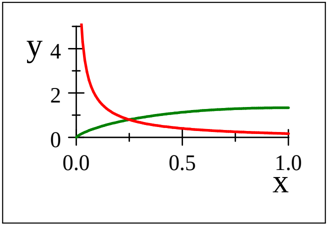

Figure 1. Set of graphs of the p.d.f. for different values of the parameters (σ,λ,c): green $(1,2,1)$; red $(0.1,2,0.5)$.

Weighted Log-Lindley distribution - structural shape properties

Fact

For $\lambda \geq 0$ and $\sigma + c > 1$ and $\lambda(\sigma + c - 1) < 1$, f attains its maximum at $x_0 = e^{\frac{\lambda-1}{\sigma+c-1}}$.

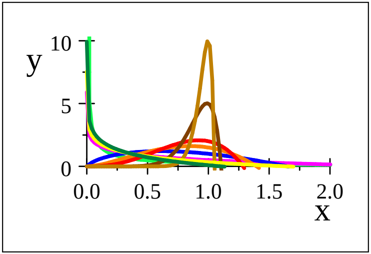

Figure 2. Set of graphs of the p.d.f. for different values of the parameters $(\sigma, \lambda, c) = (1; 1; 0.5)$ green; $(1; 1; 1)$ magenta; $(1; 1; 2)$ blue; $(1; 1; 3)$ orange; $(1; 0.1; 4)$red; $(1; 0.1; 6)$ brown; $(1; 0.1; 8)$ siena.

Weighted Log-Lindley distribution - structural shape properties

Fact

For $\lambda \geq 0$ and $\sigma + c > 1$ and $\lambda(\sigma + c - 1) > 1$, f is decreasing.

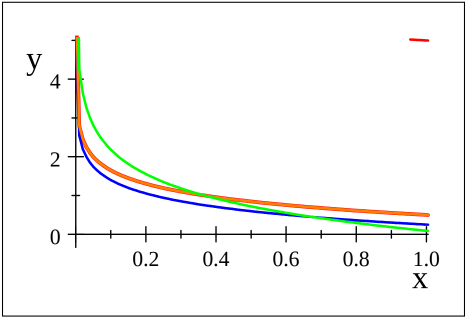

Figure 3. Set of graphs of the p.d.f. for different values of the parameters $(\sigma, \lambda, c) = (0.5; 2, 2)$ blue; $(0.1; 10; 10)$ orange; $(0.01; 10; 100)$ green.

The hazard rate

$$r(x) = \frac{f_X(x)}{S_X(x)} = \frac{(\sigma + c)^2 (\lambda - \ln x)}{x\left[(\sigma + c)\log x - (1 + \lambda(\sigma + c))\, x^{-(\sigma + c)}\right]}$$

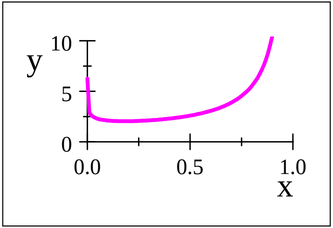

Figure 4. The hazard rate function for different values of its parameters: $(\sigma, \lambda, c) = (0.1; 1; 1)$.

- The hazard rate funtion has U-shape and an asymptote at 1 for all values of σ and λ.
- For a fixed value of λ and $0 < \sigma \leq 1$, there is also an asymptote at 0.

The Shannon entropy

Proposition. $\mathsf{E}(\ln X) = -\dfrac{2+\lambda(\sigma+c)}{(\sigma+c)[1+\lambda(\sigma+c)]}$.

$\mathsf{E}(\ln(\lambda - \ln X)) =$
$\dfrac{1}{1+\lambda(\sigma+c)}\Big[(1+\lambda(\sigma+c))\ln x + 1 - e^{\lambda(\sigma+c)}\,\mathrm{Ei}(\lambda(\sigma+c))\Big]$, where
$\mathrm{Ei}(z) = -\int\limits_{z}^{\infty}\dfrac{e^{-\omega}}{\omega}\mathrm{d}\omega$ is the exponential integral function.

Proposition. The Shannon entropy of the Weighted Log-Lindley distribution is given by:

$$\mathsf{H}(X) = \lg\frac{1}{1+\lambda(\sigma+c)} + \frac{1}{1+\lambda(\sigma+c)}\Big[(1+\lambda(\sigma+c))\ln x + 1 - e^{\lambda(\sigma+c)}\,\mathrm{Ei}(\lambda(\sigma+c))\Big] - \frac{(\sigma+c-1)[\lambda(\sigma+c)+2]}{(\sigma+c)[1+\lambda(\sigma+c)]}$$

The Shannon entropy

Theorem

The p.d.f. of a r.v. $X \sim WLL(\sigma, \lambda, \alpha)$ is the unique solution of the optimization problem:

$$f = \arg\max \quad H(g)$$

$$E_g(\ln X) = -\frac{2 + \lambda(\sigma + c)}{(\sigma + c)[1 + \lambda(\sigma + c)]}$$

$$E_g(\ln(\lambda - \alpha \ln X)) = \frac{1}{1 + \lambda(\sigma + c)}\Big[(1 + \lambda(\sigma + c))\ln x + 1$$

Parameter estimation: method of moments

$$\begin{cases} \dfrac{(\sigma+c)^2}{1+\lambda(\sigma+c)} \cdot \dfrac{1+\lambda(\sigma+c+1)}{(\sigma+c+1)^2} = \overline{x}_1 \\[2ex] \dfrac{(\sigma+c)^2}{1+\lambda(\sigma+c)} \cdot \dfrac{1+\lambda(\sigma+c+2)}{(\sigma+c+2)^2} = \overline{x}_2 \\[2ex] \dfrac{(\sigma+c)^2}{1+\lambda(\sigma+c)} \cdot \dfrac{1+\lambda(\sigma+c+3)}{(\sigma+c+3)^2} = \overline{x}_3 \end{cases}$$

Likelihood inference

$$L(x_1, x_2, ..., x_n; \sigma, \lambda, c) = \frac{(\sigma + c)^{2n}}{[1 + \lambda(\sigma + c)]^n} \prod_{i=1}^{n} (\lambda - \ln x_i) \prod_{i=1}^{n} x_i^{\sigma + c - 1}$$

$$l(x_1, x_2, ..., x_n; \sigma, \lambda, \alpha) =$$

$$2n \ln(\sigma + c) - n \ln[1 + \lambda(\sigma + c)] + \sum_{i=1}^{n} \ln (\lambda - \ln x_i) + (\sigma + c - 1) \sum_{i=1}^{n} \ln x_i$$

$$\begin{cases} \dfrac{\partial l}{\partial \sigma} = \dfrac{2n}{\sigma + c} - \dfrac{n\lambda}{1 + \lambda(\sigma + c)} + \sum_{i=1}^{n} \ln x_i = 0 \\ \dfrac{\partial l}{\partial \lambda} = -\dfrac{n(\sigma + c)}{1 + \lambda(\sigma + c)} + \sum_{i=1}^{n} \dfrac{1}{\lambda - \ln x_i} = 0 \\ \dfrac{\partial l}{\partial c} = \dfrac{2n}{\sigma + c} - \dfrac{n\lambda}{1 + \lambda(\sigma + c)} + \sum_{i=1}^{n} \ln x_i = 0 \end{cases}$$

Conclusions

- The Weighted Log-Lindley distribution, a new generalization of the Lindley distribution and an extension of the Log-Lindley distribution was considered.
- The Mathematical properties, reliability analysis, statistical estimation and stochastic ordering were investigated.
- The hazard rate function can have various shapes, so this approach is more realistic and provides a greater degree of flexibility.
- The reliability behavior of the Weighted Log-Lindley distribution allows an improved performance for lifetime data modeling.

References

E. Gomez-Deniz, M.A. Sordo and E. Calderin-Ojeda. The Log-Lindley distribution as an alternative to the beta regression model with applications in insurance, Insurance Math. Econom. 54, 49-57, 2014.

Ghitany, M.E., Al-Mutairi, D.K., 2009. Estimation methods for the discrete Poisson–Lindley distribution. Journal of Statistical Computation and Simulation 79 (1), 1–9.

Ghitany, M.E., Atieh, B., Nadarajah, S., 2008. Lindley distribution and its application. Mathematics and Computers in Simulation 78, 493–506.

Gómez-Déniz, E., Calderín-Ojeda, E., 2011. The discrete Lindley distribution: properties and applications. Journal of Statistical Computation and Simulation 81 (11), 1405–1416.

Lindley, D.V., 1958. Fiducial distributions and Bayes's theorem. Journal of the Royal Statistical Society: Series B 20 (1), 102–107.

Acknowledgements

This paper has been financially supported within the project entitled Programe doctorale si postdoctorale - suport pentru cresterea competitivitatii cercetarii în domeniul Stiintelor exacte, contract number POSDRU/159/1.5/S/ 137750. This project is co-financed by European Social Fund through Sectoral Operational Programme for Human Resources Development 2007-2013. Investing in people!

Thank you for your attention!