On the estimation of the central core of a set. Algorithms to estimate the λ-medial axis

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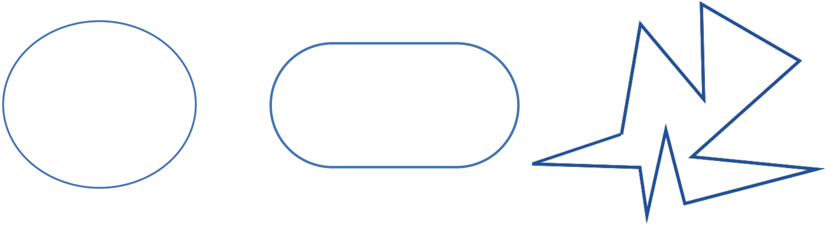

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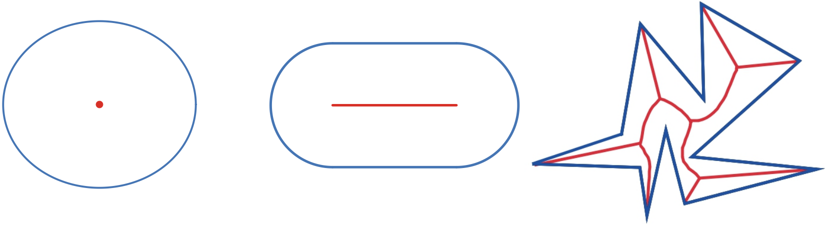

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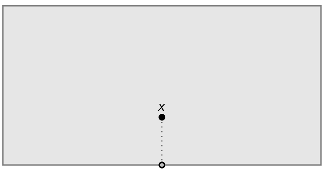
Medial axis

- Let C be a bounded set in $\mathbb{R}^d$ with non-empty interior such that $C = \overline{\text{int}(C)}$.
- For any $x \in C$, let us denote by $\Gamma(x)$ the set of boundary points closest to x,

$$\Gamma(x) = \{y \in \partial C : d(x, y) = d(x, \partial C)\},$$

where $d(x, y) = ||x - y||$ denotes the Euclidean distance between x and y in $\mathbb{R}^d$ and also for a set $A \subset \mathbb{R}^d$, $d(x, A) = \inf\{d(x, y) : y \in A\}$.



Medial axis

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The **medial axis** of C, $\mathcal{M}(C)$, is the set of points x of C, that have at least two closest boundary points, that is, $\mathcal{M}(C) = \{x \in C : |\Gamma(x)| \geq 2\}$ where $|E|$ denotes the cardinal of E.

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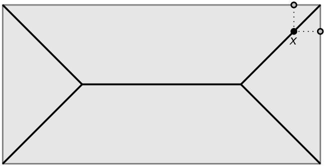

λ-medial axis

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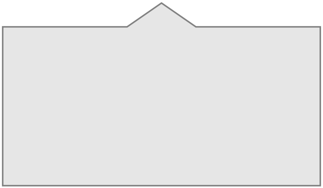

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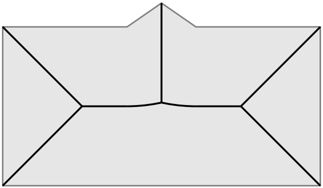
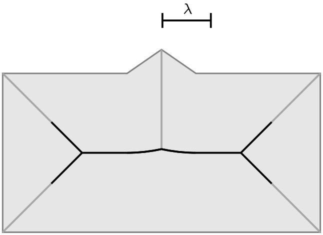


λ-medial axis

λ-medial axis [Chazal and Lieutuer (2005)]

The **λ-medial axis** of C, $\mathcal{M}_\lambda(C)$, is the set

$$\mathcal{M}_\lambda(C) = \{x \in C : \text{ for every ball } B(y,r), \text{ such that } \Gamma(x) \subset B(y,r) \text{ we have } r \geq \lambda\}.$$

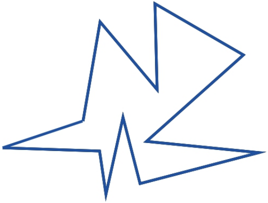


The λ-medial axis leaves out those points of the medial axis whose projections on C are too close together.

Chazal, F. and Lieutier, A. (2005).
The "λ-medial Axis". *J. Graphical Models*, 67, 304–331.

Estimation of the λ-medial axis

- This work deals with the statistical problem of estimating the λ-medial axis, $\mathcal{M}_\lambda(C)$ from a random sample of points $X_1, \ldots, X_n$ drawn inside C.



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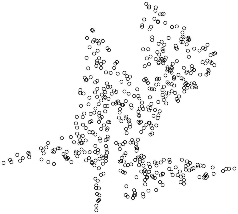

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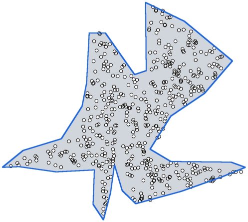

- The whole approach relies on a simple plug-in idea: we will use methods of set estimation to get a suitable estimator $C_n = C_n(X_1, \ldots, X_n)$ of C and approximate $\mathcal{M}_\lambda(C)$ by means of $\mathcal{M}_\lambda(C_n)$.
- If we choose an estimator $C_n = C_n(X_1, \ldots, X_n)$ of C such that $d_H(C_n, C) \to 0$ and $d_H(\partial C_n, \partial C) \to 0$ a.s, then $\mathcal{M}_\lambda(C_n)$ is a d_H-consistent estimator of $\mathcal{M}_\lambda(C)$.

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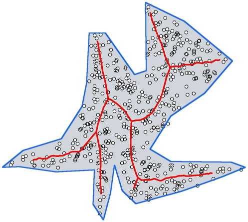

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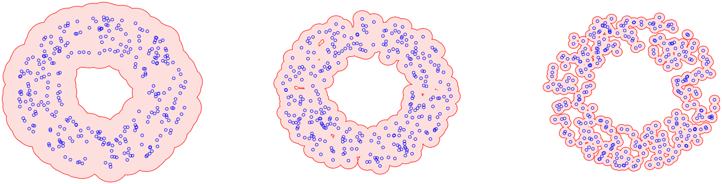
Applications to estimation. The Devroye-Wise estimator

- Given $\mathcal{X}_n = \{X_1, \ldots, X_n\}$ on a compact set C, the Devroye-Wise estimator is

$$C_n = \bigcup_{i=1}^{n} B(X_i, \epsilon_n),$$

where ϵ_n is a sequence of smoothing parameters.

- The boundary ∂C_n of the Devroye-Wise estimator consistently estimates ∂C, whenever the sequence $\epsilon_n \to 0$ is chosen large enough to ensure that $C \subset C_n$.



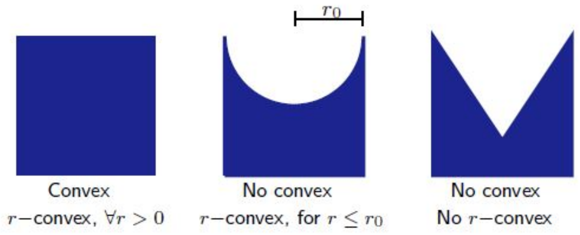
Cuevas, A. and Rodríguez-Casal, A. (2004).
On Boundary Estimation *Adv. Appl. Prob.*, 36, 340–354.

Applications to estimation. The r-convex hull estimator

- Let us assume that C fulfils the so-called r-convexity.
- A closed set $C \subset \mathbb{R}^d$ is said to be r-convex, $r > 0$, if for any $x \notin C$ there exist an open ball with radius r, B_x, such that

$$x \in B_x, \quad B_x \cap C = \emptyset,$$

that is, x can be separated from C by an open ball with radius r.

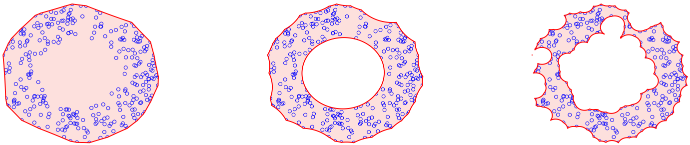


Applications to estimation. The r-convex hull estimator

- By analogy with the convex hull, we may define the r-convex hull of a set A as the "minimal r-convex set containing A", that is

$$\mathrm{Conv}_r(A) = \bigcap_{\mathrm{int}(B(x,r))\cap A=\emptyset} \left(\mathrm{int}(B(x,r))\right)^c$$

- The r-convex hull $C_n = \mathrm{Conv}_r(\mathcal{X}_n)$ of a sample $\mathcal{X}_n = \{X_1, \ldots, X_n\}$, drawn on a compact support C, provides a natural estimator for C.



Rodríguez-Casal, A. (2007)

Set estimation under convexity type assumptions. *Annales de l'I.H.P. Probabilités & Statistiques*, 43, 763-774.

Computational issues

- The exact computation of the medial axis is a difficult task, even in $\mathbb{R}^2$.
- Research has long focused on the computation of the medial axis of simple polygons. In that case, the medial axis is formed by straight-line segments and parabolic arcs.
- Attali and Montanvert (1997) characterize the medial axis of a union of balls and Amenta and Kolluri (2001) present an algorithm for its construction.
- Not much has been published so far about the exact computation of the λ-medial axis of a set.
- We propose algorithms to compute the exact λ-medial axis of sets whose shape is given by a union of balls (such as the Devroye-Wise estimator) or by the complement of a union of balls (such as the r-convex hull estimator).

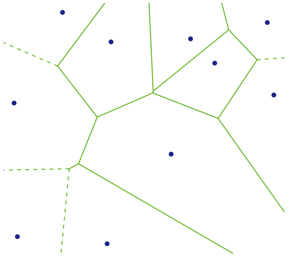
Algorithms to compute the λ-medial axis

- Given a finite set of points in the plane $\mathcal{C} = \{c_1, \ldots, c_k\}$, the Voronoi diagram of $\mathcal{C}$, $\mathrm{Vor}(\mathcal{C})$, is defined as a family of regions (Voronoi cells)

$$V_i = \{x \in \mathbb{R}^2 : \; d(x, c_i) \leq d(x, c_j), \forall j = 1, \ldots, k\}, \; i = 1, \ldots, k$$

(V_i is the set of points closest to c_i than to any other point in $\mathcal{C}$)

- We will denote by $\mathrm{Vor}_0(\mathcal{C})$ the union of the Voronoi edges of $\mathrm{Vor}(\mathcal{C})$.



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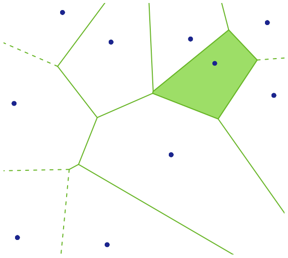

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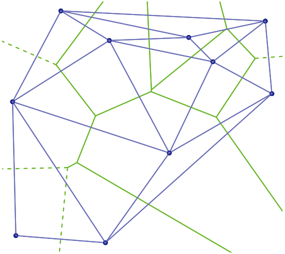
- The Delaunay triangulation of the set of points $\mathcal{C}$, Del($\mathcal{C}$), is defined as the straight line dual of the Voronoi diagram Vor($\mathcal{C}$).
- Each Voronoi cell V_i corresponds to the Delaunay vertex c_i.
- The Delaunay triangulation contains a straight line edge between the Delaunay vertices c_i and c_j if and only if V_i and V_j share a common edge. Therefore, each Voronoi edge corresponds to its dual Delaunay edge.
- Finally, each Voronoi vertex corresponds to a Delaunay triangle.



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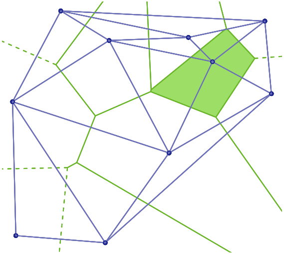
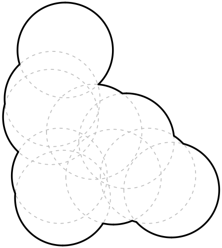


An algorithm to compute the λ-medial axis of the Devroye-Wise estimator

- Let C_n be the Devroye-Wise estimator of a sample $\mathcal{X}_n = \{X_1, \ldots, X_n\}$ in $\mathbb{R}^2$.
- The exact medial axis of C_n can be obtained using the algorithm by Amenta and Kolluri (2001).
- This algorithm computes the medial axis of a union of balls from its dual shape.

An algorithm to compute the λ-medial axis of the Devroye-Wise estimator

- Let $\mathcal{U}$ denote the union of a set of balls in $\mathbb{R}^2$ with equal radii.
- Let $\text{Vor}(\mathcal{C})$ be the Voronoi diagram of their centers and $\text{Del}(\mathcal{C})$ the corresponding Delaunay triangulation.
- The dual shape $\mathcal{S}$ of the union of balls $\mathcal{U}$ is the union of all the Delaunay simplices (vertices, edges and triangles) for which their corresponding Voronoi duals (cells, edges and vertices, respectively) intersect $\mathcal{U}$.



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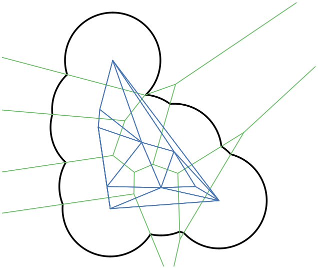

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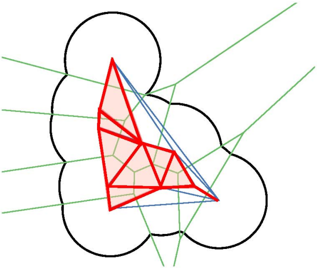
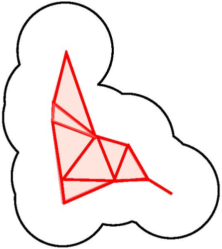


An algorithm to compute the λ-medial axis of the Devroye-Wise estimator

- Note that $\mathcal{S}$ could include isolated points or edges which are not part of any triangle in $\mathcal{S}$. These points and edges are the so-called singular faces. The regular components are the connected components in $\mathcal{S}$ that remain after removing the singular faces.



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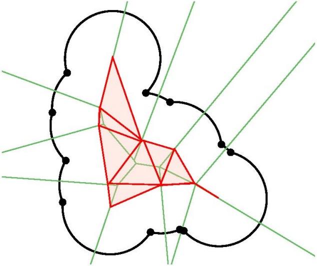
- Given $\mathcal{U}$, a union of balls in $\mathbb{R}^2$, let us denote by $\mathcal{V}$ the set of vertices of $\mathcal{U}$, that is, the points in $\partial\mathcal{U}$ contained in the intersection of the balls in $\mathcal{U}$.
- According to Theorem 2 in Amenta and Kolluri (2001), given a union of balls, $\mathcal{U}$, and its dual shape, $\mathcal{S}$, the medial axis of $\mathcal{U}$ consists of the singular faces of $\mathcal{S}$ plus the subset of $\mathrm{Vor}_0(\mathcal{V})$ which intersects the regular components of $\mathcal{S}$.



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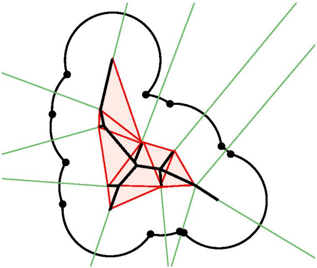

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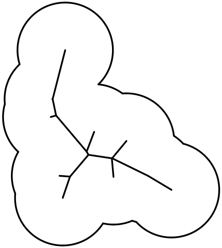

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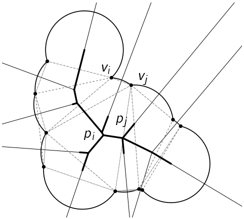
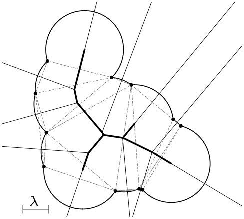
Algorithm 1. Compute the λ-medial axis of the Devroye-Wise estimator

1. Given $\mathcal{X}_n$, compute the Devroye-Wise estimator C_n.
2. Compute $\mathcal{M}(C_n)$ using the algorithm by Amenta and Kolluri (2001).
3. Initialize $\mathcal{M}_\lambda(C_n)$ to $\mathcal{M}(C_n)$.
4. Let $\mathcal{V}$ be the set of vertices of C_n. For each edge e in $\mathcal{M}(C_n)$ let v_i and v_j in $\mathcal{V}$ defining the dual edge of e in the Delaunay triangulation of $\mathcal{V}$. If the distance between v_i and v_j is lower than 2λ, then the edge e is completely removed from $\mathcal{M}_\lambda(C_n)$. Otherwise, e belongs to $\mathcal{M}_\lambda(C_n)$.





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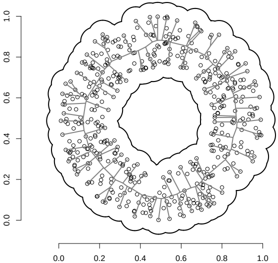
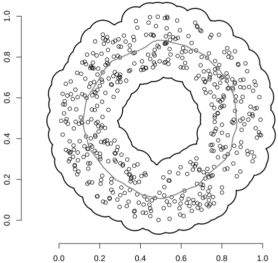




An algorithm to compute the λ-medial axis of the r-convex hull estimator

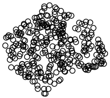
- Let C_n be the r-convex hull of a sample $\mathcal{X}_n = \{X_1, \ldots, X_n\}$ in $\mathbb{R}^2$.
- We propose a procedure (Algorithm r-hull 1) to compute the exact medial axis of C_n and then, we adapt this algorithm to calculate the λ-medial axis of C_n (Algorithm r-hull 2).
- The boundary of C_n consists of the union of a finite number of circumference arcs of radius r. Therefore, C_n is contained in the complement of a finite union of equal-radius balls.
- By Proposition 4.5 in Attali (1995), the medial axis of the complement of a finite union of equal-radius balls can be obtained from the corresponding Voronoi diagram of their centers.

Skip

An algorithm to compute the λ-medial axis of the r-convex hull estimator

Algorithm r-hull 1. Computing the medial axis of the r-convex hull estimator

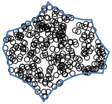
1. Compute C_n, the r-convex hull of $\mathcal{X}_n$.
2. Determine the centers $\mathcal{C} = \{c_i, i = 1, \ldots, k\}$ of the circumference arcs of radius r that define the boundary of C_n.
3. Compute $\mathrm{Vor}(\mathcal{C})$.
4. Return $\mathcal{M}(C_n) = \mathrm{Vor}_0(\mathcal{C}) \cap C_n$, the intersection of C_n with the edges of $\mathrm{Vor}(\mathcal{C})$.



An algorithm to compute the λ-medial axis of the r-convex hull estimator

Algorithm r-hull 1. Computing the medial axis of the r-convex hull estimator

1. Compute C_n, the r-convex hull of $\mathcal{X}_n$.
2. Determine the centers $\mathcal{C} = \{c_i, i = 1, \ldots, k\}$ of the circumference arcs of radius r that define the boundary of C_n.
3. Compute $\mathrm{Vor}(\mathcal{C})$.
4. Return $\mathcal{M}(C_n) = \mathrm{Vor}_0(\mathcal{C}) \cap C_n$, the intersection of C_n with the edges of $\mathrm{Vor}(\mathcal{C})$.



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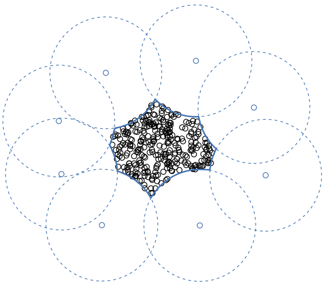

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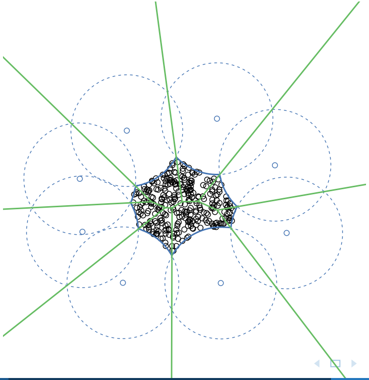

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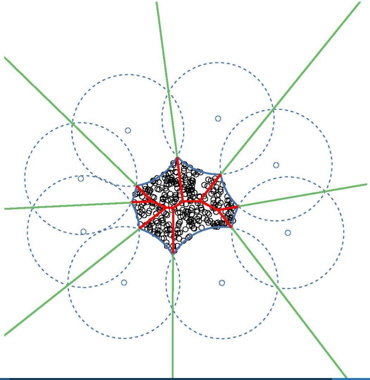

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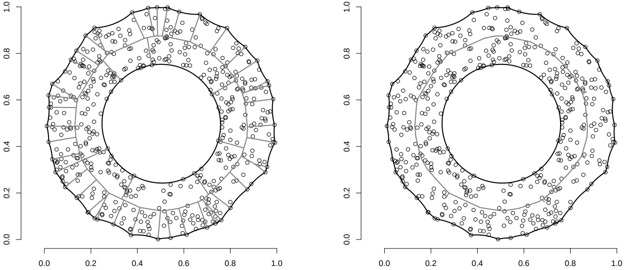
Algorithm r-hull 2. Compute the λ-medial axis of the r-convex hull estimator

1. Compute C_n, the r-convex hull of $\mathcal{X}_n$.
2. Compute $\mathcal{M}(C_n)$ using Algorithm 1.
3. Initialize $\mathcal{M}_\lambda(C_n)$ to $\mathcal{M}(C_n)$.
4. For each edge e in $\mathcal{M}(C_n)$ with vertices p_i and p_j, let c_i and c_j be the centers in $\mathcal{C}$ defining the dual edge of e in the Delaunay triangulation.

   Case 1. One of the vertices of e belongs to ∂C_n
   Case 2. Both vertices of e belong to the interior of C_n



Examples

- We analyze the behavior of our algorithms with an example of shape stored in binary image format file.
- The data have been rescaled to the unit square.
- We have obtained a uniform sample of size $n = 4000$ from the image.
- The algorithms have been implemented in R.

Examples

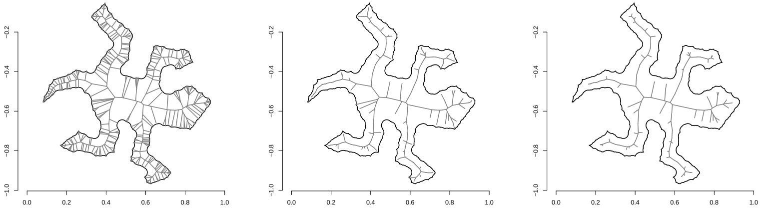


Figure: In solid black line boundary of C_n, the r-convex hull estimator for $r = 0.025$. (Left) In solid gray line, $\mathcal{M}(C_n)$. (Middle) In solid gray line, $\mathcal{M}_\lambda(C_n)$ for $\lambda = 0.01$. (Right) In solid gray line, $\mathcal{M}_\lambda(C_n)$ for $\lambda = 0.03$.

Examples

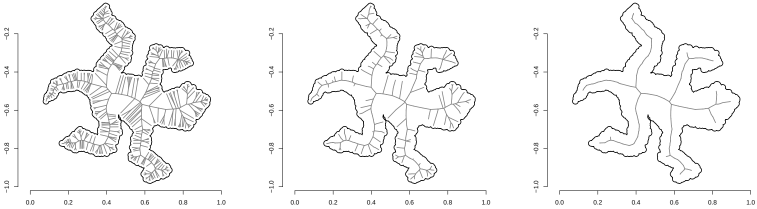


Figure: In solid black line boundary of C_n, the Devroye-Wise estimator for $\epsilon = 0.015$. (Left) In solid gray line, $\mathcal{M}(C_n)$. (Middle) In solid gray line, $\mathcal{M}_\lambda(C_n)$ for $\lambda = 0.01$. (Right) In solid gray line, $\mathcal{M}_\lambda(C_n)$ for $\lambda = 0.03$.

Cuevas, A., Llop, P., Pateiro-López, B. (2014).
On the estimation of the medial axis and inner parallel body. *Journal of Multivariate Analysis*, **129**, 171-185.