Estimation of the offspring mean matrix in 2-type critical Galton-Watson processes

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The model: single-type process

Definition

The process $(X_n)_{n\in\mathbb{N}}$ is a (single-type) Galton–Watson process if

$$X_n = \sum_{k=1}^{X_{n-1}} \xi_{n,k},$$

where $(\xi_{n,k})_{n,k\in\mathbb{N}}$ consists of i.i.d. nonnegative, integer valued random variables, namely the offspring variables of an individual;

We can suppose $X_0 = 1$, since starting from any initial value $k > 1$ the resulting Galton–Watson process is just the sum of k independent Galton–Watson processes each starting from 1.

Extinction

Let $m_\xi := \mathbb{E}(\xi_{1,1})$ denote the offspring mean. We can distinguish 3 cases based on the value of m_ξ:

- If $m_\xi < 1$, the the process is subcritical
- If $m_\xi = 1$, the the process is critical
- If $m_\xi > 1$, the the process is supercritical

The extinction theorem

Let $(X_n)_{n\in\mathbb{N}}$ be a Galton–Watson process. If the process is subcritical or critical and $\mathrm{P}(\xi_{1,1} = 1) < 1$, then extinction happens almost surely, that is

$$\mathrm{P}(\exists n \in \mathbb{N}, X_n = 0) = 1.$$

If the process is supercritical, then

$$\mathrm{P}(\exists n \in \mathbb{N}, X_n = 0) < 1.$$

The model: introducing immigration

Definition

The process $(X_n)_{n\in\mathbb{N}}$ is a (single-type) Galton–Watson process with immigration if

$$X_n = \sum_{k=1}^{X_{n-1}} \xi_{n,k} + \varepsilon_n,$$

where

- $(\xi_{n,k})_{n,k\in\mathbb{N}}$ consists of i.i.d. nonnegative, integer valued random variables, namely the offspring variables of an individual;
- $(\varepsilon_n)_{n\in\mathbb{N}}$ are i.i.d. nonnegative, integer valued random variables, namely the immigration variables;
- the offspring and immigration variables are independent from each other.

For the sake of simplicity, we suppose $X_0 = 0$.

The parameters

We retain the notation for m_ξ and the classification based on its value. Let $m_\varepsilon := \mathbb{E}(\varepsilon_1)$ denote the immigration mean, while let

$$C_\xi := \mathrm{Var}(\xi_{1,1}), \qquad C_\varepsilon := \mathrm{Var}(\varepsilon_1)$$

denote the offspring and immigration variances respectively.

Our aim is to estimate the value of m_ξ, based on the sample

$$X_1, X_2, \dots, X_n$$

while we suppose to know the values of $m_\varepsilon, C_\xi, C_\varepsilon$.

Estimation: condtional least squares method

We use the conditional least squres method to obtain an estimate. Introduce the martingale differences

$$M_k := X_k - \mathbb{E}(X_k | X_{k-1}) = X_k - m_\xi X_{k-1} - m_\varepsilon.$$

If we minimize the sum

$$\sum_{k=1}^{n} |M_k|^2$$

with respect to m_ξ we get the so called conditional least squares estimate

$$\widehat{m}_\xi = \sum_{k=1}^{n} (X_k - m_\varepsilon) X_{k-1} \left(\sum_{k=1}^{n} X_{k-1}^2 \right)^{-1}.$$

We want to describe the asymptotic properties of the estimator in the critical case.

Previously known results

The following is a functional limit theorem for single-type critical Galton–Watson processes.

Theorem (Wei, Winnicki 1989)

Suppose $(X_n)_{n\in\mathbb{N}}$ is a Galton-Watson process with immigration, that is critical, hence $m_\xi = 1$, and $X_0 = 0$, $m_\varepsilon \neq 0$, $E(\xi_{1,1}^2) < \infty$, $E(\varepsilon_1^2) < \infty$. Then

$$\mathcal{X}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{X} \qquad \text{as } \ n \to \infty,$$

where $\mathcal{X}_t^{(n)} = n^{-1} X_{\lfloor nt \rfloor}, \ t \geq 0$ and $\mathcal{X}$ is the unique strong solution of the stochastic differential equation

$$d\mathcal{X}_t = m_\varepsilon dt + \sqrt{C_\xi \mathcal{X}_t^+} d\mathcal{W}_t, \quad \mathcal{X}_0 = 0,$$

where $(\mathcal{W}_t)_{t\geq 0}$ is a standard Wiener process.

Previously known results

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Suppose $(X_n)_{n\in\mathbb{N}}$ is a Galton-Watson process with immigration, that is critical, hence $m_\xi = 1$, and $X_0 = 0$, $m_\varepsilon \neq 0$, $E(\xi_{1,1}^2) < \infty$, $E(\varepsilon_1^2) < \infty$. Then

$$n(\widehat{m}_\xi - 1) \xrightarrow{\mathcal{D}} \frac{\int_0^1 \mathcal{X}_t^2 \, d(\mathcal{X}_t - m_\varepsilon t)}{\int_0^1 \mathcal{X}_t^2 \, dt} \qquad \text{as } \; n \to \infty.$$

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The model

Definition

The process $(X_n)_{n\in\mathbb{N}}$ is a 2-type Galton–Watson process with immigration if

$$X_n = \sum_{k=1}^{X_{n-1,1}} \xi_{n,k,1} + \sum_{k=1}^{X_{n-1,2}} \xi_{n,k,2} + \varepsilon_n ,$$

where

- $(\xi_{n,k,i})_{n,k\in\mathbb{N}}$ consists of i.i.d. nonnegative, integer valued random vectors for $i = 1, 2$, namely the offspring vectors of a type i individual;
- $(\varepsilon_n)_{n\in\mathbb{N}}$ are i.i.d. nonnegative, integer valued random vectors, namely the immigration vectors;
- the offspring and immigration vectors are independent from each other.

For the sake of simplicity, we suppose $X_0 = 0$.

Parameters

Let

$$m_\xi := \begin{bmatrix}\mathbb{E}(\xi_{1,1,1}) & \mathbb{E}(\xi_{1,1,2})\end{bmatrix} = \begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix}, \qquad m_\varepsilon := \mathbb{E}(\varepsilon_1),$$

denote the offspring mean matrix and the immigration mean vector, and

$$C_i := \mathrm{Var}(\xi_{1,1,i}), \qquad C_\varepsilon := \mathrm{Var}(\varepsilon_1)$$

denote the offspring variances and the immigration variance.

We distinguish 3 cases based on ϱ, the spectral radius of m_ξ.

- If $\varrho < 1$, then the process is subcritical.
- If $\varrho = 1$, then the process is critical.
- If $\varrho > 1$, then the process is supercritical.

If we assume criticality, the the eigenvalues of m_ξ are 1 and $\alpha + \delta - 1$.

Eigenvectors of m_ξ

Let

$$u_{\text{left}} := \frac{1}{2-\alpha-\delta}\begin{bmatrix}\gamma+1-\delta\\ \beta+1-\alpha\end{bmatrix}, \quad u_{\text{right}} := \frac{1}{\beta+1-\alpha}\begin{bmatrix}\beta\\ 1-\alpha\end{bmatrix}$$

and

$$v_{\text{left}} := \frac{1}{\beta+1-\alpha}\begin{bmatrix}-(1-\alpha)\\ \beta\end{bmatrix}, \quad v_{\text{right}} := \frac{1}{2-\alpha-\delta}\begin{bmatrix}-(\beta+1-\alpha)\\ \gamma+1-\delta\end{bmatrix}$$

denote left and right eigenvectors of the eigenvalues 1 and $\alpha+\delta-1$ respectively.
Then the powers of the matrix m_ξ behaves the following way

$$m_\xi^k = u_{\text{right}}u_{\text{left}}^\top + (\alpha+\delta-1)^k v_{\text{right}}v_{\text{left}}^\top, \quad k \in \mathbb{N}.$$

A limit theorem for the process

Let

$$C_\xi := \frac{\beta C_1 + (1-\alpha) C_2}{\beta + 1 - \alpha}.$$

Theorem (Ispány & Pap 2012)

Let $(X_n)_{n\in\mathbb{N}}$ be a critical, 2-type Galton-Watson process with immigration. Suppose $X_0 = 0$, $m_\varepsilon \neq 0$, and $\mathbb{E}(\|\xi_{1,1,i}\|^2) < \infty$, $\mathbb{E}(\|\varepsilon_1\|^2) < \infty$. Then

$$(n^{-1} X_{\lfloor nt \rfloor})_{t\geq 0} \xrightarrow{\mathcal{D}} (\mathcal{Y}_t u_{\mathrm{right}})_{t\geq 0} \qquad \text{as} \quad n \to \infty,$$

where $\mathcal{Y}$ is the unique strong solution of the SDE

$$d\mathcal{Y}_t = \langle u_{\mathrm{left}}, m_\varepsilon \rangle dt + \sqrt{\langle C_\xi u_{\mathrm{left}}, u_{\mathrm{left}} \rangle \mathcal{Y}_t^+} \, d\mathcal{W}_t, \quad \mathcal{Y}_0 = 0,$$

and $(\mathcal{W}_t)_{t\geq 0}$ is a standard Wiener process.

CLS estimator

Introduce the following martingale differences

$$M_k := X_k - \mathbb{E}(X_k | X_{k-1}) = X_k - m_\xi X_{k-1} - m_\varepsilon .$$

If we minimize the sum

$$\sum_{k=1}^{n} \|M_k\|^2$$

with respect to m_ξ we get the so called conditional least squares estimate

$$\widehat{m}_\xi = \sum_{k=1}^{n} (X_k - m_\varepsilon) X_{k-1}^\top \left(\sum_{k=1}^{n} X_{k-1} X_{k-1}^\top \right)^{-1} = B_n A_n^{-1}.$$

Alternative form for the estimate

By the continous mapping theorem

$$n^{-3}A_n = \int_0^1 n^{-2} X_{\lfloor ns \rfloor} X_{\lfloor ns \rfloor}^\top \, ds \xrightarrow{\mathcal{D}} \int_0^1 \mathcal{Y}_s^2 \, ds \, u_{\mathrm{left}} u_{\mathrm{left}}^\top .$$

Since $u_{\mathrm{left}} u_{\mathrm{left}}^\top$ is singular we have to find another form for handling our estimate.

Expressing the inverse of A_n with it's adjugate matrix before finding the limit helps

$$\widehat{m}_\xi = \frac{B_n \operatorname{adj}(A_n)}{\det(A_n)}$$

Rewriting the process in terms of eigenvectors

Define

$$U_k := \langle u_{\text{left}}, X_k \rangle, \quad V_k := \langle v_{\text{left}}, X_k \rangle.$$

Then

$$\begin{aligned} X_k &= U_k u_{\text{right}} + V_k v_{\text{right}} \\ U_k &= U_{k-1} + \langle u_{\text{left}}, M_k \rangle + \langle u_{\text{left}}, m_\varepsilon \rangle, \\ V_k &= (\alpha + \delta - 1) V_{k-1} + \langle v_{\text{left}}, M_k \rangle + \langle v_{\text{left}}, m_\varepsilon \rangle. \end{aligned}$$

Using the continous mapping theorem yields

$$n^{-(\ell+1)} \sum_{k=1}^{n} U_{k-1}^{\ell} \xrightarrow{\mathcal{D}} \int_0^1 \langle u_{\text{left}}, \mathcal{Y}_t u_{\text{right}} \rangle^{\ell} \, dt = \int_0^1 \mathcal{Y}_t^{\ell} \, dt,$$

$$n^{-(\ell+1)} \sum_{k=1}^{n} V_{k-1}^{\ell} \xrightarrow{\mathcal{D}} \int_0^1 \langle v_{\text{left}}, \mathcal{Y}_t u_{\text{right}} \rangle^{\ell} \, dt = 0.$$

Rewriting $\det(A_n)$ in terms of eigenvectors

Using the variables U_k and V_k we can express $\det(A_n)$ in the following way

$$\begin{aligned}\det(A_n) &= \sum_{k=1}^{n} X_{k-1,1}^2 \sum_{k=1}^{n} X_{k-1,2}^2 - \left(\sum_{k=1}^{n} X_{k-1,1} X_{k-1,2}\right)^2 \\ &= \sum_{k=1}^{n} U_{k-1}^2 \sum_{k=1}^{n} V_{k-1}^2 - \left(\sum_{k=1}^{n} U_{k-1} V_{k-1}\right)^2 .\end{aligned}$$

We need to determine the asymptotics of

$$\sum_{k=1}^{n} V_{k-1}^2, \qquad \sum_{k=1}^{n} U_{k-1} V_{k-1}.$$

A limit theorem for the estimator

Theorem (Körmendi & Pap 2014)

Let $(X_n)_{n\in\mathbb{N}}$ be a critical, 2-type Galton-Watson process with immigration. Suppose $X_0 = 0$, $m_\varepsilon \neq 0$, and $\mathbb{E}(\|\xi_{1,1,i}\|^8) < \infty$, $\mathbb{E}(\|\varepsilon_1\|^8) < \infty$. If $\langle C_\xi v_{\mathrm{left}}, v_{\mathrm{left}}\rangle > 0$, then

$$\sqrt{n}(\widehat{m}_\xi - m_\xi) \xrightarrow{\mathcal{D}} \sqrt{1-(\alpha+\delta-1)^2}\,\frac{\int_0^1 \mathcal{Y}_t C_\xi^{1/2}\, d\widetilde{\mathcal{W}}_t v_{\mathrm{left}}^\top}{\langle C_\xi v_{\mathrm{left}}, v_{\mathrm{left}}\rangle^{1/2}\int_0^1 \mathcal{Y}_t\, dt}$$

as $n \to \infty$, where $\mathcal{Y}$ is the unique strong solution of the SDE

$$d\mathcal{Y}_t = \langle u_{\mathrm{left}}, m_\varepsilon\rangle dt + \sqrt{\langle C_\xi u_{\mathrm{left}}, u_{\mathrm{left}}\rangle \mathcal{Y}_t^+}\, d\mathcal{W}_t, \quad \mathcal{Y}_0 = 0,$$

and $(\widetilde{\mathcal{W}}_t)_{t\geq 0}$ is a 2-dimensional Wiener process independent from $\mathcal{W}$.

A limit theorem for the estimator

Since the estimator for m_ξ is weakly consistent we can define an estimator for the criticality parameter ϱ as well.

Theorem (Körmendi & Pap 2014)

Let $(X_n)_{n\in\mathbb{N}}$ be a critical, 2-type Galton-Watson process with immigration. Suppose $X_0 = 0$, $m_\varepsilon \neq 0$, and $\mathbb{E}(\|\xi_{1,1,i}\|^8) < \infty$, $\mathbb{E}(\|\varepsilon_1\|^8) < \infty$. If $\langle C_\xi v_{\mathrm{left}}, v_{\mathrm{left}}\rangle > 0$, then

$$n(\widehat{\varrho} - 1) \xrightarrow{\mathcal{D}} \frac{\int_0^1 \mathcal{Y}_t \, d\left(\mathcal{Y}_t - t\langle u_{\mathrm{left}}, m_\varepsilon\rangle\right)}{\int_0^1 \mathcal{Y}_t^2 \, dt}$$

as $n \to \infty$.

Possible generalizations

1. Continous time branching processes.

   We are working on this, but only have results with heavy restrictions on the structure of the equivalent of m_ξ.

2. Increasing the number of types to general d-type GWI processes.

   If $d > 2$ then we don't have a natural estimator for the criticality parameter ϱ, eigenvalues with multiplicity greater than 1 may cause problems.

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