Asymptotic Theory for Copula Rank-Based Periodograms

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joint work with:
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Three Time Series Models, $X_t = Y_t / \text{Var}(Y_t)^{1/2}$

- QAR(1) process, Koenker and Xiao (2006),

$$Y_t = 0.1\Phi^{-1}(U_t) + 1.9(U_t - 0.5)Y_{t-1}$$

  - (U_t) i. i. d. standard uniform random variables,
  - Φ the distribution function of the standard normal distribution.

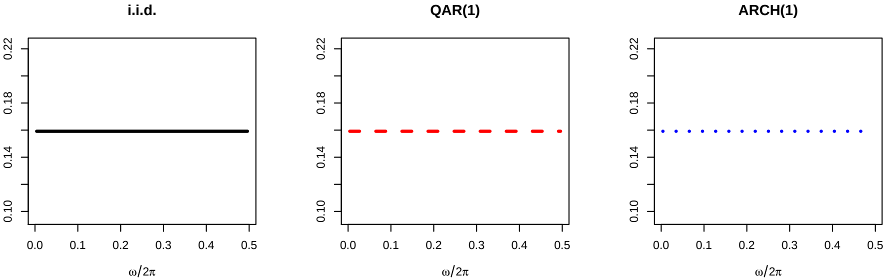
- ARCH(1) process, Engle (1982),

$$Y_t = \left(1/1.9 + 0.9Y_{t-1}^2\right)^{1/2} \varepsilon_t$$

  - (ε_t) standard normal white noise.

- Independent Gaussian white noise.

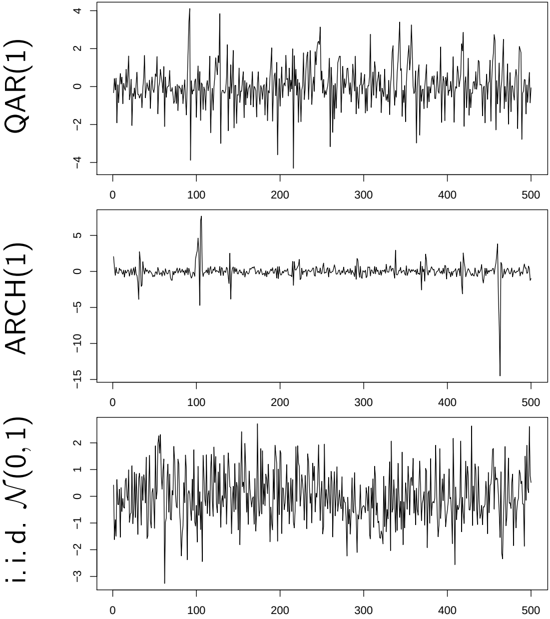
Spectral densities

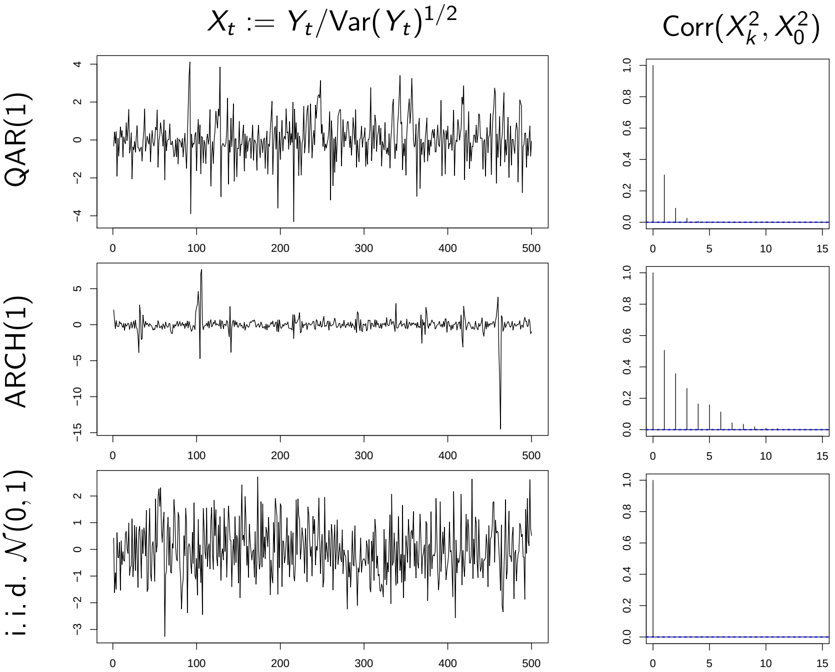


Spectral density:

$$\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_t, X_{t-k})e^{-ik\omega}$$

$$X_t := Y_t / \mathrm{Var}(Y_t)^{1/2}$$

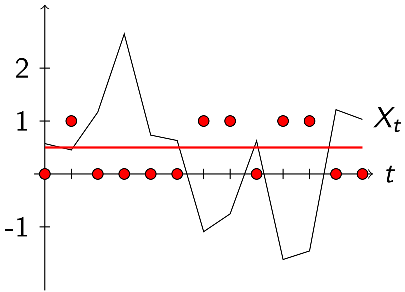




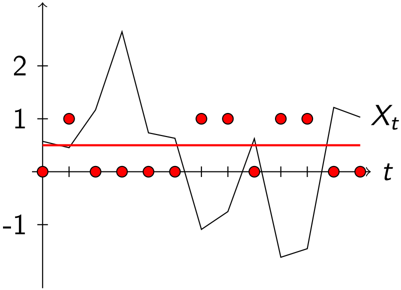
$X_t := Y_t/\mathrm{Var}(Y_t)^{1/2}$
$\mathrm{Corr}(X_k^2, X_0^2)$
QAR(1)
ARCH(1)
i.i.d. $\mathcal{N}(0,1)$
0
100
200
300
400
500
0
5
10
15

Outline

Clipped Process $(I\{X_t \leq 0.5\})_{t\in\mathbb{N}}$



Clipped Processes
$(I\{X_t \leq 0.5\})_{t\in\mathbb{N}}$ and $(I\{X_t \leq -1\})_{t\in\mathbb{N}}$



A Quantile-Based Measure for Serial Dependence

(X_t) stationary

Traditionally: Auto-covariances of lag k:

$$\mathrm{Cov}(X_t, X_{t-k})$$

Analysis of the spectral density

$$\mathfrak{f}(\omega) := \sum_{k=-\infty}^{\infty} \mathrm{Cov}(X_t, X_{t-k}) \mathrm{e}^{-\mathrm{i}k\omega}$$

is analysis of the auto-covariances.

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A Quantile-Based Measure for Serial Dependence

(X_t) strictly stationary, F cdf of X_t

New approach: Copula cross-covariances of lag k:

$$\gamma_k(\tau_1, \tau_2) := \mathrm{Cov}(I\{F(X_t) \leq \tau_1\}, I\{F(X_{t-k}) \leq \tau_2\})$$

Analysis of the Copula spectral density kernel

$$\mathfrak{f}^{\tau_1,\tau_2}(\omega) := \sum_{k=-\infty}^{\infty} \mathrm{Cov}(I\{F(X_t) \leq \tau_1\}, I\{F(X_{t-k}) \leq \tau_2\}) \mathrm{e}^{-\mathrm{i}k\omega}$$

is analysis of the copula cross-covariances.

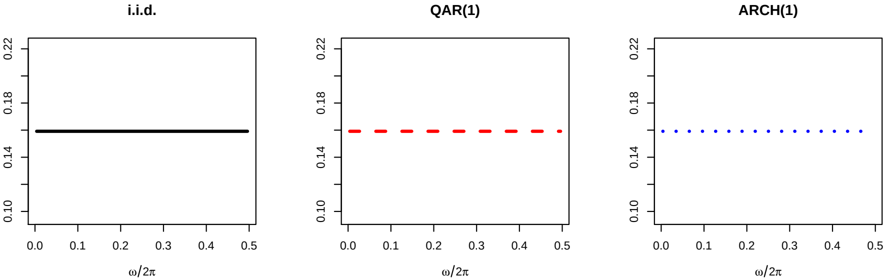
Covariance and *Cross-covariance*

- $\gamma_k(\tau_1, \tau_2)$ is cross-covariance of bivariate time series

$$(I\{F(X_t) \leq \tau_1\}, I\{F(X_t) \leq \tau_2\}),$$

- $\gamma_k(\tau_1, \tau_2)$ always exist (no assumptions about moments),
- $\gamma_k(\tau_1, \tau_2) = \mathbb{P}(F(X_t) \leq \tau_1, F(X_{t-k}) \leq \tau_2) - \tau_1\tau_2$
  $\Rightarrow$ Copula: disentangling serial and marginal features
- Invariance of γ_k under continuous monotone transformation
- $\{\gamma_k(\tau_1, \tau_2) | \ \tau_1, \tau_2 \in (0,1)\}$ and F entirely characterize the joint distribution of (X_t, X_{t-k}),
- ... if $\mathbb{E}X_t^2 < \infty$, then this includes the acf of $(X_t)_{t \in \mathbb{Z}}$,

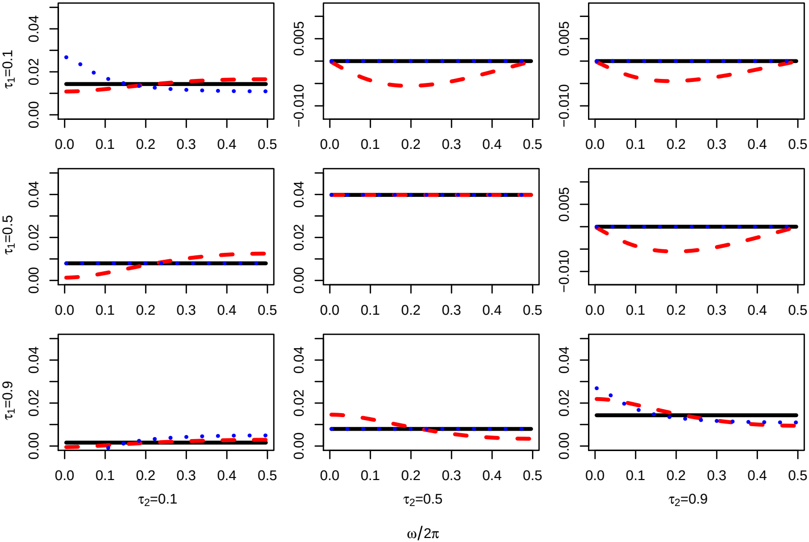
Spectral densities



Spectral density:

$$\frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \mathrm{Cov}(X_t, X_{t-k}) \mathrm{e}^{-\mathrm{i}k\omega}$$

Copula spectral density kernels



Modifying the (traditional) Periodogram

From $X_0, \ldots, X_{n-1}$, for $\omega_j := \frac{2\pi j}{n} \in (0, \pi)$, $j \in \mathbb{Z}$, compute

$$I_n(\omega_j) := \frac{1}{2\pi n} d_n(\omega_j) d_n(-\omega_j),$$

where

$$d_n(\omega_j) := \sum_{t=0}^{n-1} X_t \mathrm{e}^{-\mathrm{i}t\omega_j}$$

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where

$$d^{\tau}_{n,U}(\omega_j) := \sum_{t=0}^{n-1} I\{F \quad (X_t) \leq \tau\} \mathrm{e}^{-\mathrm{i} t \omega_j}$$

Modifying the (traditional) Periodogram

From $X_0, \ldots, X_{n-1}$, for $\omega_j := \frac{2\pi j}{n} \in (0, \pi)$, $j \in \mathbb{Z}$, compute

$$I_{n,R}(\omega_j) := \frac{1}{2\pi n} d^{\tau_1}_{n,R}(\omega_j) d^{\tau_2}_{n,R}(-\omega_j),$$

where

$$d^{\tau}_{n,R}(\omega_j) := \sum_{t=0}^{n-1} I\{\hat{F}_n(X_t) \leq \tau\} \mathrm{e}^{-\mathrm{i}t\omega_j}$$

$$\hat{F}_n(x) := \frac{1}{n} \sum_{t=0}^{n-1} I\{X_t \leq x\}$$

Asymptotic properties of $I_{n,R}^{\tau_1,\tau_2}(\omega)$

Theorem (K., Volgushev, Dette, Hallin (2015+))

Under suitable technical assumptions

$$I_{n,R}^{\cdot,\cdot}(\omega) \rightsquigarrow \mathbb{I}^{\cdot,\cdot}(\omega) \quad \text{in } \ell^\infty([0,1]^2) \quad \omega \in (0,\pi)$$

where $\mathbb{I}^{\tau_1,\tau_2}(\omega) := (2\pi)^{-1}\mathbb{D}^{\tau_1}(\omega)\overline{\mathbb{D}^{\tau_2}(\omega)}$, $\mathbb{D}$ *a centered, complex-valued Gaussian process with covariance structure*

$$\mathrm{Cov}(\mathbb{D}^{\tau_1}(\omega_1), \mathbb{D}^{\tau_2}(\omega_2)) = 2\pi\mathfrak{f}^{\tau_1,\tau_2}(\omega_1)I\{\omega_1 = \omega_2\}.$$

In particular: $\mathbb{I}^{\cdot,\cdot}(\omega_1), \mathbb{I}^{\cdot,\cdot}(\omega_2)$ *independent for* $\omega_1 \neq \omega_2$ *and* $\mathbb{E}[\mathbb{I}^{\tau_1,\tau_2}(\omega)] = \mathfrak{f}^{\tau_1,\tau_2}(\omega)$.

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In particular: $\mathbb{I}^{\cdot,\cdot}(\omega_1), \mathbb{I}^{\cdot,\cdot}(\omega_2)$ *independent for* $\omega_1 \neq \omega_2$ *and* $\mathbb{E}[\mathbb{I}^{\tau_1,\tau_2}(\omega)] = \mathfrak{f}^{\tau_1,\tau_2}(\omega)$.

Smoothing the CR periodogram kernel

- 'Asymptotic expectation' of $I_{n,R}^{\tau_1,\tau_2}(\omega)$ correct, no consistency
- Classical approach: smoothing

Definition (The smoothed CR periodogram kernel)

For kernel W and bandwidth b_n, let

$$\hat{\mathfrak{f}}_{n,R}^{\tau_1,\tau_2}(\omega) := \frac{2\pi}{n} \sum_{s=1}^{n-1} W_n\big(\omega - 2\pi s/n\big) I_{n,R}^{\tau_1,\tau_2}(2\pi s/n),$$

where $W_n(u) := \sum_{j=-\infty}^{\infty} b_n^{-1} W(b_n^{-1}[u + 2\pi j]).$

Asymptotic properties of $\hat{\mathfrak{f}}_{n,R}^{\tau_1,\tau_2}(\omega)$

Theorem (K., Volgushev, Dette, Hallin (2015+))

Under suitable assumptions, for any fixed $\omega \in (0,\pi)$

$$\mathbb{G}_n(\cdot,\cdot;\omega) := \sqrt{nb_n}\Big(\hat{\mathfrak{f}}_{n,R}^{\tau_1,\tau_2}(\omega) - \mathfrak{f}^{\tau_1,\tau_2}(\omega) - B_n^{(k)}(\tau_1,\tau_2;\omega)\Big)_{\tau_1,\tau_2\in[0,1]} \rightsquigarrow \mathbb{H}(\cdot,\cdot;\omega)$$

in $\ell^\infty([0,1]^2)$*, where* $B_n^{(k)}(\tau_1,\tau_2;\omega) := \sum_{j=1}^{k} \frac{b_n^j}{j!} \int v^j W(v)dv \frac{d^j}{d\omega^j}\mathfrak{f}^{\tau_1,\tau_2}(\omega)$*, and* $\mathbb{H}(\cdot,\cdot;\omega)$ *is a centered, complex-valued Gaussian process with*

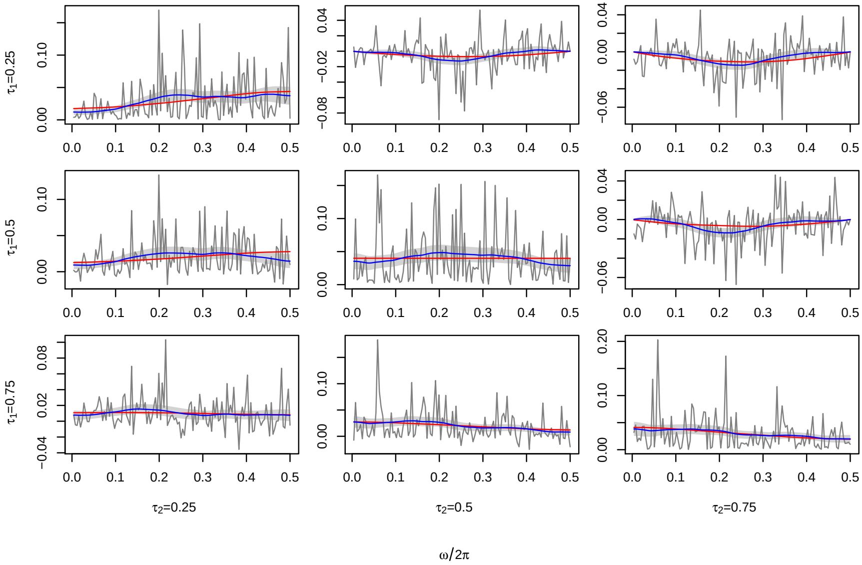
$$\mathrm{Cov}\left(\mathbb{H}(x_1,y_1;\omega),\mathbb{H}(x_2,y_2,\omega)\right) = \mathfrak{f}^{x_1,y_1}(\omega)\mathfrak{f}^{x_2,y_2}(\omega)\int W^2(u)du.$$

Moreover $\mathbb{H}(\omega) = \overline{\mathbb{H}(-\omega)} = \mathbb{H}(\omega+2\pi)$ *and* $\{\mathbb{H}(\omega), \omega \in [0,\pi]\}$ *is a family of independent processes. In particular, the weak convergence above holds jointly for finite, fixed collections of frequencies* ω.

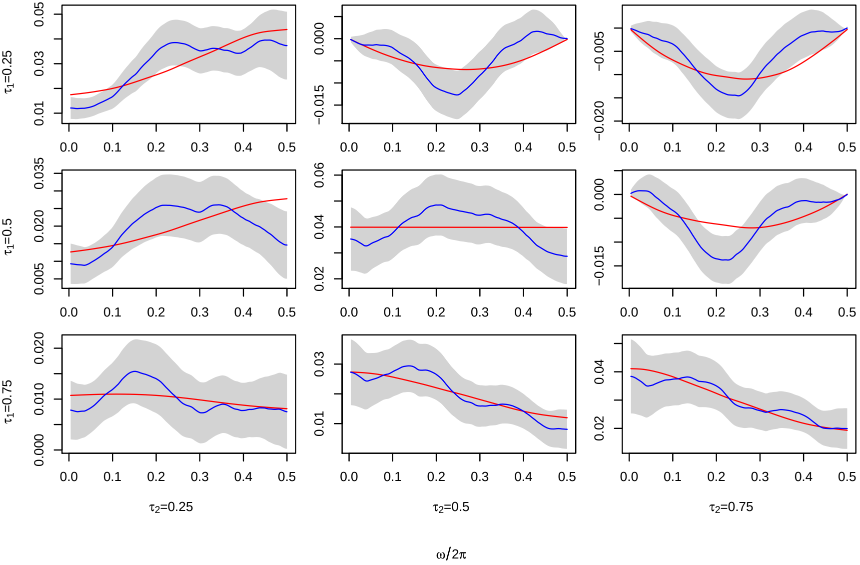
Example using the R-package quantspec

```r
library(quantspec)

Y <- ts3(256)

levels <- c(0.25, 0.5, 0.75)
FF <- 2 * pi * (0:128) / 256
K <- length(levels)

wgt <- kernelWeight(W = W1, b = 0.1)

sPG.cl <- smoothedPG(Y, levels.1 = levels,
  type = "clipped", weight = wgt)

sCSD <- quantileSD(N = 2^9, type = "copula", ts = ts3,
seed.init = 2581, levels.1 = levels, R = 1000)

plot(sPG.cl, plotPG = TRUE, qsd = sCSD, ratio = 1.7,
  frequencies = FF[FF > 0], type.CIs = "naive.sd",
  type.scaling = "individual")
```

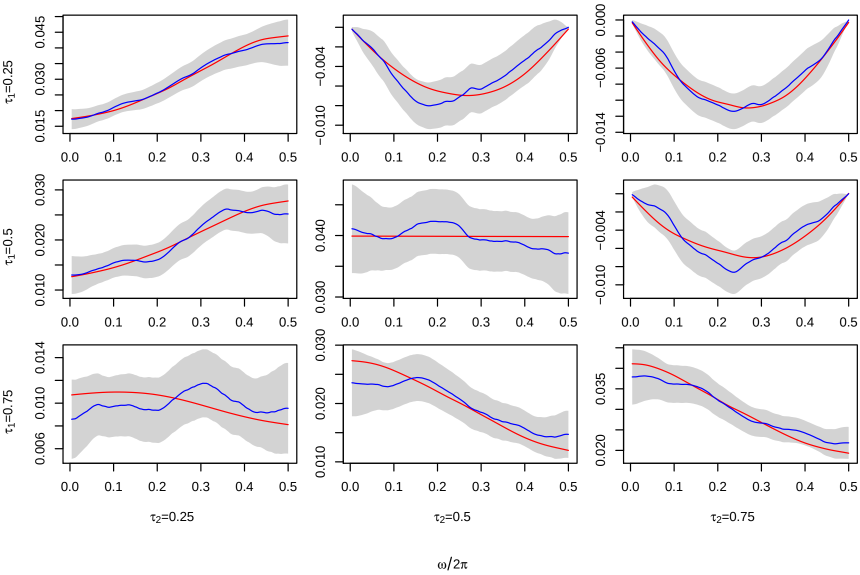
Plot of the SmoothedPG in the example, $n = 256$



Plot of the SmoothedPG in the example, $n = 256$



Plot of the SmoothedPG in the example, $n = 1.024$



“Model-free” and “nonlinear” spectral analysis

Quantile-based measures of serial dependence:

- Separation of serial dependencies and marginal features,
- Invariance under monotone transformations.

Quantile-based periodograms:

- inherit many of the properties of the ordinary periodogram,
- Robustness can be expected,
- Analysis of pair-copulae, not simply covariances,
- Weak convergence in $\ell^\infty([0,1]^2)$,
- no linearity, distributional, nor even moment assumptions required.

Much work remains on the Research Agenda

- Tests based on the CR periodogram kernel,
- Estimation of higher-order spectra,
- Estimation of integrated spectra,
- Bootstrap,
- Locally stationary processes,
- ...

References

- Dette, H., Hallin, M., Kley, T. and Volgushev, S. (2015). Of Copulas, Quantiles, Ranks and Spectra: an L_1-approach to spectral analysis. *Bernoulli*, Vol. 21(2), 781-831.
- Kley, T., Volgushev, S., Dette, H. and Hallin, M. (2015+). Quantile Spectral Processes: Asymptotic Analysis and Inference. *Bernoulli*, forthcoming. Available on `http://arxiv.org/abs/1401.8104`.
- Kley, T. (2015). quantspec: Quantile-based Spectral Analysis Functions. R package version 1.0-3. Available on `http://cran.r-project.org/web/packages/quantspec/index.html`.