Convergence in distribution in metric and submetric spaces

19th European Young Statisticians Meeting
Prague, 31 August - 4 September 2015

Adam Jakubowski
Nicolaus Copernicus University
Toruń, Poland

Convergence in distribution

- A commonly accepted definition, valid for arbitrary topological space $(\mathcal{X}, \tau)$ is

$$X_n \underset{\mathcal{D}}{\longrightarrow} X_0 \text{ iff } \mathbb{E}f(X_n) \to \mathbb{E}f(X_0),$$

  for every bounded and τ-continuous function $f : \mathcal{X} \to \mathbb{R}^1$.

- Equivalently, if $\mu_n \sim X_n$, $n = 0, 1, 2, \ldots$, then for every bounded and τ-continuous function f

$$\int_{\mathcal{X}} f(x)\, d\mu_n(x) \to \int_{\mathcal{X}} f(x)\, d\mu_0(x).$$

- In other words, converegnce in distribution of random elements is identified with weak-$*$ convergence of distributions.

The purpose of the talk

- We want to motivate and introduce a new definition of the notion of convergence in distribution of random elements with tight laws.
- This new definition coincides with the usual one on metric spaces and spaces of distributions (like $\mathcal{S}', \mathcal{D}'$).
- The advantage is that the new definition allows us to preserve the whole power of the metric theory in a wide class of topological spaces called submetric spaces.
- The theory brings a new light even in the case of metric spaces, by showing that the crucial property is rather the shape of compact sets and not the completeness.

The purpose of the talk

- The tools and results of the theory are presently used in the areas of stochastic partial differential equations, stochastic analysis and mathematical finance.
- Why to bother with non-metric spaces?
  Strong (metric) topologies are suitable for approximation!
  Weak (non-metric) topologies are useful in existence problems!

Submetric spaces

- A topological space $(\mathcal{X}, \tau)$ is said to be submetric, if there exists a separately continuous metric on $\mathcal{X}$.
- Warning: The topology generated by such a metric is, in general, coarser than the original topology.
- The most common example of a submetric space is a topological space $(\mathcal{X}, \tau)$ on which there exist a countable family $\{f_i\}_{i \in \mathbb{I}}$ of τ-continuous functions, which separate points of $\mathcal{X}$, i.e. if $f_i(x) = f_i(y)$ for all $i \in \mathbb{I}$, then $x = y$.
- $$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|f_i(x) - f_i(y)|}{1 + |f_i(x) - f_i(y)|}.$$

Example: Weak topology on a Hilbert space

Let us begin with the simplest non-metric topology frequently used in mathematics.

- Let $(\mathbb{H}, < \cdot, \cdot >)$ be a real, separable, infinite dimensional Hilbert space.
- Let $\tau_w = \sigma(\mathbb{H}, \mathbb{H})$ be the weak topology on $\mathbb{H}$, i.e. the coarsest topology with respect to which all linear functionals of the form $< \cdot, y >$ are continuous.
- $(\mathbb{H}, \tau_w)$ is submetric!
- For, let $\{y_i\}_{i \in \mathbb{I}}$ be a countable dense subset of $\mathbb{H}$. We set $f_i(x) =< x, y_i >$.
- It is known that $\mathcal{B}_{\tau_w} = \mathcal{B}_{\|\cdot\|} = \sigma\{f_i\,;\, i \in \mathbb{I}\}$.

An example due to Fernique (1967)

- Le $\{e_j\}_{j=0,1,2\dots}$ be an orthonormal basis in $\mathbb{H}$.
- Let $a_n \nearrow 1$ so fast that $\lim_{n\to\infty} n^2 \log a_n = 0$.
- Set $X_n = ne_j$ with probability $(1 - a_n)a_n^j$, $j = 0, 1, 2, \dots$.
- Then for every $y \in \mathbb{H}$

$$< X_n, y > \underset{\mathcal{P}}{\longrightarrow} 0 =< 0, y >,$$

and so

$$X_n \underset{\mathcal{D}(\tau_w)}{\longrightarrow} 0.$$

- But for each $K > 0$

$$\lim_{n\to\infty} P(\|X_n\| > K) = 1,$$

hence no subsequence of $\{X_n\}$ is uniformly τ_w-tight.

An example due to Fernique (1967)

- It is inconsistent with our intuitions: if $x_n \to x_0$ weakly, then $\sup_n \|x_n\| < +\infty$.
- In particular, as we shall see soon, $\{X_n\}$ dos not admit any a.s. Skorokhod representation.
- In fact, we can say more: since no subsequence of $\{X_n\}$ is uniformly τ_w-tight, so no subsequence of $\{X_n\}$ admits an a.s. Skorokhod's representation.

The a.s. Skorokhod representation for sequences

- Suppose we are given the a.s. Skorokhod representation for $\{X_n\}$, i.e. there exist $Y_0, Y_1, Y_2, \ldots$ defined on $([0,1], \mathcal{B}_{[0,1]}, \ell)$ and with values in $\mathbb{H}$ such that
  - $Y_n \sim X_n, \quad n = 0, 1, 2, \ldots,.$
  - $Y_k(\omega) \to_{\tau_w} Y_0(\omega),\ \omega \in [0,1].$
- First consequence:

$$\sup_n \| Y_n(\omega) \| < +\infty,\ \omega \in [0,1].$$

- Hence we have strong tightness of $\{Y_n\}$: for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon = \{x\,;\, \|x\| \leqslant R_\varepsilon\}$ such that

$$P(\{\omega\,;\ Y_n(\omega) \in K_\varepsilon,\ n = 1, 2, \ldots\}) > 1 - \varepsilon.$$

- This implies uniform tightness of $\{X_n\}$.

The a.s. Skorokhod representation for subsequences

- Let us continue the previous considerations, i.e. $Y_0, Y_1, Y_2, \ldots$ are defined on $([0,1], \mathcal{B}_{[0,1]}, \ell)$ and with values in $\mathbb{H}$ and are such that
  - $Y_n \sim X_n, \quad n = 0, 1, 2, \ldots,.$
  - $Y_k(\omega) \to_{\tau_w} Y_0(\omega),\ \omega \in [0,1].$
- Then clearly $g(Y_n) \to g(Y_0)$ a.s. for every sequentially continuous $g : (\mathbb{H}, \tau_w) \to \mathbb{R}^1$.
- Hence $Eh(g(X_n)) = Eh(g(Y_n)) \to Eh(g(Y_0)) = Eh(g(X_0))$ for every bounded and continuous $h : \mathbb{R}^1 \to \mathbb{R}^1$ and so $g(X_n) \ \longrightarrow_{\mathcal{D}}\ g(X_0)$.
- In fact, to get $g(X_n) \ \longrightarrow_{\mathcal{D}}\ g(X_0)$ we do not need the a.s. Skorokhod representation for the whole sequence.
- We need to be able in every subsequence $\{X_{n_k}\}$ find a further subsequence $\{X_{n_{k_l}}\}$ which admits the a.s. Skorokhod representation.

A characterization of convergence in distribution on metric spaces

- Let $(\mathcal{X}, \rho)$ be a metric space and $\mathcal{P}(\mathcal{X}, \rho)$ be the space of tight laws on $\mathcal{X}$.
- Clearly, if $X_n \longrightarrow_{\mathcal{P}} X_0$, then $\mu_n \Rightarrow \mu_0$.
- Hence any mapping of the form

$$L_0\Big((\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{X}, \rho)\Big) \ni X \mapsto \mathbb{P} \circ X^{-1} \in \mathcal{P}(\mathcal{X}, \rho)$$

is continuous, when $L_0\Big((\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{X}, \rho)\Big)$ is equipped with the metric topology of convergence in probability.

Theorem

The sequential topology $\tau(\Rightarrow)$ (of weak convergence) on $\mathcal{P}(\mathcal{X}, \rho)$ is the finest topology with this property.

The question: is it possible to transfer this characterization to submetric spaces?

A universal characterization

Thorem

Let $(\mathcal{X}, \tau)$ be a submetric space. There exists a convergence $\Rrightarrow$ on $\mathcal{P}(\mathcal{X}, \tau)$ such that the sequential topology $\tau(\Rrightarrow)$ on $\mathcal{P}(\mathcal{X}, \tau)$ is the finest topology for which every embedding

$$L_0\Big((\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{X}, \tau)\Big) \ni X \mapsto \mathbb{P} \circ X^{-1} \in \mathcal{P}(\mathcal{X}, \tau)$$

is continuous, when $L_0\Big((\Omega, \mathcal{F}, \mathbb{P}), (\mathcal{X}, \rho)\Big)$ is equipped with the sequential topology of strongly tight almost sure convergence.

In particular, if $(\mathcal{X}, \rho)$ is a metric space, then $\tau(\Rrightarrow)$ and $\tau(\Rightarrow)$ coincide on $\mathcal{P}(\mathcal{X}, \rho)$.

In fact, on metric spaces the convergencies $\Rightarrow$ and $\stackrel{*}{\Rrightarrow}$ are identical.

Complements on sequential topologies

- In general consider an $\mathcal{L}$-convergence $x_n \longrightarrow x_0$.
    - x_0 is determined uniquely;
    - If $x_n \equiv x_0$, $n \in \mathbb{N}$, then $x_n \longrightarrow x_0$;
    - If $x_n \longrightarrow x_0$ and $\{x_{n_k}\}$ is a subsequence, then also $x_{n_k} \longrightarrow x_0$;
- $\mathcal{L}$-convergence generates the sequential topology $\tau(\rightarrow)$ given by the familiar recipe

Closed sets

$F \subset \mathcal{X}$ is $\tau(\rightarrow)$-closed if F contains all limits of $\longrightarrow$-convergent sequences of elements of F.

- The topology $\tau(\rightarrow)$ determines another convergence, so-called $\mathcal{L}^*$-convergence, which is $\mathcal{L}$-convergence and satisfies additionally
    - If in every subsequence $\{x_{n_k}\}$ one can find a further subsequence $\{x_{n_{k_l}}\}$ such that $x_{n_{k_l}} \rightarrow_{\tau(\rightarrow)} x_0$, then the whole sequence $x_n \rightarrow_{\tau(\rightarrow)} x_0$.

Complements on sequential topologies

- The convergence $\longrightarrow_{\tau(\longrightarrow)}$ (called “a posteriori") is in general weaker than the original (=“a priori") convergence $\longrightarrow$. How much weaker?

The Kantorovich-Vulikh-Pinsker-Kisyński (KVPK) Recipe

x_n converges to x_0 a posteriori iff every subsequence $x_{n_1}, x_{n_2}, \ldots$ of $\{x_n\}$ contains a further subsequence $x_{n_{k_1}}, x_{n_{k_2}}, \ldots$ convergent to x_0 a priori.

- Instead of writing $\longrightarrow_{\tau(\longrightarrow)}$ we will use the notation $\xrightarrow{*}$.
- An example: the topology on $L^0(\Omega, \mathcal{F}, P)$ generated by the a.s. convergence.
- Another example: metric convergence.
- Another example: metric convergence at given rate.

Sequential topology generated by a topology

Definition

Let $(\mathcal{X}, \tau)$ be a Hausdorff topological space. Say that $F \subset \mathcal{X}$ is τ_s-closed if limits of τ-convergent sequences of elements of F remain in F, i.e. if $x_n \in F,\ n = 1, 2, \ldots$ and $x_n \longrightarrow_{\tau} x_0$, then $x_0 \in F$. The topology given by τ_s-closed sets is called the sequential topology generated by τ and will be denoted by τ_s.

Theorem

Let $(\mathcal{X}, \tau)$ be a Hausdorff topological space. Then

- $\tau \subset \tau_s$ (i.e. τ_s is finer than τ).
- $x_n \longrightarrow_{\tau_s} x_0$ if and only if $x_n \longrightarrow_{\tau} x_0$.

In particular, τ_s is Hausdorff.

Why submetric spaces?

Theorem

Let $(\mathcal{X}, \tau)$ be a submetric space. Then $K \subset \mathcal{X}$ is τ-compact if, and only if, K is sequentially τ-compact.

Theorem

Let $(\mathcal{X}, \tau)$ be a submetric space. The sequential topology τ_s is the finest topology on $\mathcal{X}$ among topologies with the same compact sets as τ.

Uniform τ-tightness implies uniform τ_s-tightness

Convergence in distribution in submetric spaces

Here we provide the promised definition of the convergence in distribution of random elements with values in submetric spaces and with tight laws. We will write for this new notion $X_n \xrightarrow{**}_{\mathcal{D}} X_0$ or $\mu_n \overset{*}{\Rightarrow} \mu_0$.

Definition

$X_n \xrightarrow{**}_{\mathcal{D}} X_0$ if every subsequence $\{n_k\}$ contains a sub-subsequence $\{n_{k_l}\}$ such that $\{X_{n_{k_l}} : l = 1, 2, \ldots\}$ and X_0 admit a Skorokhod representation $\{Y_l\}$ defined on the Lebesgue interval, which is almost surely convergent and strongly tight.

Recall that the last statement means that Y_n converges to Y_0 a.s. in the usual sense and for each $\varepsilon >$ there exists a τ-compact subset $K_\varepsilon \subset \mathcal{X}$ such that

$$P(\{\omega \in [0,1] : Y_l(\omega) \in K_\varepsilon, l = 1, 2, \ldots\}) > 1 - \varepsilon.$$

Convergence in distribution in submetric spaces

- We have already proved that $X_n \xrightarrow{**}_{\mathcal{D}} X_0$ implies $X_n \longrightarrow_{\mathcal{D}} X_0$.
- Is the definition operational?

The strong version of the Direct Prohorov Theorem in submetric spaces (AJ 1997)

If $\{\mu_i\}_{i\in\mathbb{I}}$ is a uniformly tight family of probability measures on a submetric space $(\mathcal{X},\tau)$, then its every subsequence $\{\mu_n\}$ contains a further subsequence $\{\mu_{n_k}\}$ which admits a strongly tight a.s. Skorokhod representation on $[0,1]$.

The a.s. representation for subsequences on submetric spaces

Corollary - the a.s. Skorokhod representation for subsequences in submetric spaces

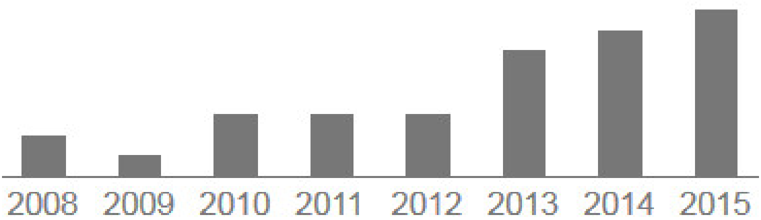
If $X_n \quad \longrightarrow_{\mathcal{D}} \; X_0$ on $(\mathcal{X}, \tau)$ and $\{X_n\}$ is uniformly tight, then in each subsequence $\{X_{n_k}\}_{k\in\mathbb{N}}$ one can find a sub-subsequence $\{X_{n_{k_l}}\}_{l\in\mathbb{N}}$ which admits the a.s. strongly tight Skorokhod representation $\{Y_l\}$ on $[0,1]$ (with $Y_0 \sim X_0$).

- There are submetric spaces, for which the a.s. Skorokhod representation does not hold for the whole sequence (Bogachev and Kolesnikov (2001), Banakh, Bogachev, Kolesnikov (2004)).
- The a.s. Skorokhod representation in submetric spaces has been successfully applied e.g. in the theory of stochastic partial differential equations.

The a.s. Skorokhod representation in submetric spaces

- M. Odreját, Stochastic nonlinear wave equations in local Sobolev spaces, Electr. J. Probab. **15 (2010)**, 1041–1091.
- Z. Brzeźniak & M. Ondrejat, Stochastic geometric wave equations with values in a compact Riemannian homogeneous spaces, Ann. Probab. **41 (2013)**, 1938–1977.
- D. Wang & H. Wang, Global existence of martingale solutions to the three-dimensional stochastic compressible Navier-Stokes equations, Differential Integral Equations, **28 (2015)**, 1105–1154.
- D. Breit, E. Feireisl & M. Hofmanová, Incompressible limit for compressible fluids with stochastic forcing, arXiv:1503.06684 (2015).

The a.s. Skorokhod representation in submetric spaces - citations in GS



Convergence in distribution in submetric spaces

The space $\mathcal{P}(\mathcal{X})$ of tight probability measures on $\mathcal{X}$ equipped with the sequential topology determined by $\stackrel{*}{\Rrightarrow}$ (which is finer than the usual weak-$*$ convergence) has the following remarkable properties:

- Due to the “strong version" of the Direct Prohorov Theorem the convergence $\stackrel{*}{\Rrightarrow}$ is quite operational.
- The Converse Prohorov Theorem is easy to obtain and holds in many spaces.
- No assumptions like the T_3 (regularity) property are required for the space $\mathcal{X}$ which is very important in applications to sequential spaces.
- On metric spaces and spaces of distributions (like $\mathcal{S}'$ or $\mathcal{D}'$) the theory of the usual convergence in distribution of random elements with tight probability distributions remains unchanged.

Comments - the Converse Prohorov's theorem

LeCam's Theorem (1957)

If $\mu_n \Rightarrow \mu_0$ on a metric space $\mathcal{X}$, and both all μ_n's are tight and μ_0 is tight, then $\{\mu_n\}_{n\in\mathbb{N}}$ is uniformly tight.

Prohorov's Theorem - The Converse Part (1956)

If $\mathcal{X}$ is Polish and $\{\mu_i\}_{i\in\mathbb{I}}$ is $\Rightarrow$-relatively compact, then it is also uniformly tight.

D. Preiss' Example (1973)

On rational numbers $\mathbb{Q}$ one can find a relatively compact family $\{\mu_i\}_{i\in\mathbb{I}}$, which IS NOT uniformly tight.

- For a long time it was a common belief that Preiss' example holds because of the lack of completeness.
- But it is rather because of an irregularity of compact sets in $\mathbb{Q}$!

Sequential topologies fit to the formalism of submetric spaces

- Suppose on $\mathcal{X}$ we are given an $\mathcal{L}$-convergence $\longrightarrow$. A function $f : \mathcal{X} \to \mathbb{R}^1$ is continuous with respect to $\tau(\rightarrow)$ if, and only if, it is sequentially continuous with respect to $\longrightarrow$.
- Suppose on $\mathcal{X}$ there exist a countable family $\{f_i\}$ of $\longrightarrow$-sequentially continuous functions functions, which separate points in $\mathcal{X}$.

Then $\mathcal{X}$ is a submetric space.

Example. The S topology on the Skorokhod space $\mathbb{D}$ - a path from criteria of compactness to topology

- A.J., A non-Skorohod topology on the Skorohod space, Electronic Journal of Probability, 2 (1997), No 4, 1-21.
- A.J., Convergence in various topologies for stochastic integrals driven by semimartingales, The Annals of Probability, 24 (1996), 2141-2153.
- A.B. Sow, R. Rhodes & É. Pardoux, Homogenization of periodic semilinear parabolic degenerate PDEs, Ann. I. H. Poincaré (AN), 26 (2009) 979 – 998.
- G. Guo, X. Tan & N. Touzi, Tightness and duality of martingale transport on the Skorokhod space, preprint arXiv:1507.01125, 2015.
- R. Balan, A. J. & S. Louhichi, Functional Convergence of Linear Processes with Heavy-Tailed Innovations, Journal of Theoretical Probability, DOI 10.1007/s10959-014-0581-9.

Genesis of the S topology

- Let $\{M_n(t)\}$ be a sequence of square integrable martingales satisfying

$$\sup_n \mathbb{E} M_n(1)^2 < +\infty.$$

  Under this natural assumption, can we say anything about distributional properties of processes M_n?
- Doob's inequality gives us

$$\sup_n \mathbb{E}\big(\sup_{t\in[0,1]} |M_n(t)|\big)^2 < +\infty.$$

- And the Doob-Snell inequality for the number of up-crossings leads to

$$\sup_n \mathbb{E} N^{a,b}(M_n) < +\infty, \quad a, b \in \mathbb{R}^1, a < b.$$

- Summarizing we have uniform tightness of random variables $\{\|M_n\|_\infty\,;\, n \in \mathbb{N}\}$ and $\{N^{a,b}(M_n)\,;\, n \in \mathbb{N}\}$, for all $a < b$.

Genesis of the S topology

- For quasimartingales similar observations were made by Meyer and Zheng (1984). They introduced the “pseudo-path topology” on the Skorokhod space $\mathbb{D}$ and a family of conditions on truncated variations which ensured relative compactness of distributions.
- Suppose $\mathbb{D}$ is equipped with the M-Z topology. Stricker (1985) showed that for the relative compactness of distributions of processes with trajectories in $\mathbb{D}$ we need, in fact, only uniform tightness of random variables $\{\|X_n\|_\infty\}$ and $\{N^{a,b}(X_n)\}$, for each pair of levels $a < b$.
- It was clear for Kurtz (1991) that such conditions give much more. But an *ad hoc* device constructed by Kurtz did not have a topological character.
- A.J. (1997, EJP) constructed on $\mathbb{D}$ a topology („the S topology"), for which Stricker’s conditions are equivalent to the uniform tightness.

Definition of the S topology - notations

- $\mathbb{D}$ denotes the Skorokhod space, i.e. a family of functions $x : [0,1] \to \mathbb{R}^1$, which are right-continuous at every $t \in [0,1)$ and admit left-limits at every $t \in (0,1]$.
- Let $\|x\|_\infty$ be the sup-norm on $\mathbb{D}$.
- For $a < b$, let $N^{a,b}$ be the number of up-crossings of levels a and b.
- For $\eta > 0$, let N_η be the number of η-oscillations on $[0,1]$.
- Let $\|v\|$ be the total variation of v:

$$\|v\| = \sup \{ |v(0)| + \sum_{i=1}^{m} |v(t_i) - v(t_{i-1})| \},$$

  where the supremum is taken over all $0 = t_0 < t_1 < \ldots < t_m = 1,\ m \in \mathbb{N}$.
- Set $\mathbb{V} = \{x \in \mathbb{D}\,;\ \|x\| < +\infty\}$.

Equivalent criteria of compactness in $\mathbb{D}$

Lemma

Let $K \subset \mathbb{D}$. Assume that

$$\sup_{x \in K} \|x\|_\infty < +\infty. \tag{1}$$

Then the following conditions are equivalent:

$$\text{For all } a < b \quad \sup_{x \in K} N^{a,b}(x) < +\infty. \tag{2}$$

$$\text{For every } \eta > 0 \quad \sup_{x \in K} N_\eta(x) < +\infty. \tag{3}$$

Equivalent criteria of compactness in $\mathbb{D}$

Lemma - continued

Moreover, any of the pairs of conditions (1)+(2) and (1)+(3) is equivalent to the following statement: For every $\varepsilon > 0$ and every $x \in K$ there exists $v_{x,\varepsilon} \in \mathbb{V}$ such that

$$\sup_{x \in K} \|x - v_{x,\varepsilon}\|_\infty \leqslant \varepsilon, \tag{4}$$

and

$$\sup_{x \in K} \|v_{x,\varepsilon}\| < +\infty. \tag{5}$$

Definition of the S topology

- We shall write $x_n \to_S x_0$ if for every $\varepsilon > 0$ one can find elements $v_{n,\varepsilon} \in \mathbb{V}$, $n = 0, 1, 2, \ldots$ which are ε-uniformly close to x_n's and weakly-$*$ convergent:

$$\|x_n - v_{n,\varepsilon}\|_\infty \leqslant \varepsilon, \qquad n = 0, 1, 2, \ldots, \tag{6}$$

$$v_{n,\varepsilon} \Rightarrow v_{0,\varepsilon}, \qquad \text{as } n \to \infty. \tag{7}$$

- Here $v_n \Rightarrow v_0$ means that

$$\int_{[0,1]} f(t)\, dv_n \to \int_{[0,1]} f(t)\, dv_0(t),$$

for each continuous function $f : [0, 1] \to \mathbb{R}^1$.

Theorem (Criterion of relative S-compactness)

Let $K \subset \mathbb{D}$. We can find in every sequence $\{x_n\}$ of elements of K a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \longrightarrow_S x_0$, as $k \to \infty$, if, and only if, one of the equivalent sets of conditions mentioned in the previous lemma is satisfied.

Convergence in the S topology

- $\longrightarrow_S$ defines a topology on $\mathbb{D}$.
- But the convergence in this topology, say $\xrightarrow{*}_S$, is weaker than $\longrightarrow_S$.
- The question is: can we provide a “compact” characterization of $\xrightarrow{*}_S$?

Compact definition of $\stackrel{*}{\longrightarrow}_S$

- Let $\mathbb{A}$ be a family of continuous functions A of finite variation ($\mathbb{A} \subset C([0,1]) \cap \mathbb{V}$), satisfying $A(0) = 0$.
- Let $A_n \in \mathbb{A}$, $n = 0, 1, 2, \ldots$. We say that $A_n \longrightarrow_\tau A_0$, if

$$\sup_{t \in [0,1]} |A_n(t) - A_0(t)| \to 0,$$

and

$$\sup_n \|A_n\| < +\infty.$$

- This is a „mixed topology" on $C([0,1]) \cap \mathbb{V}$.

Theorem

$x_n \stackrel{*}{\longrightarrow}_S x_0$ if, and only if, $x_n(1) \to x_0(1)$ and

$$\int_0^1 x_n(u)\, dA_n(u) \to \int_0^1 x_0(u)\, dA_0(u),$$

for each sequence $A_n \longrightarrow_\tau A_0$.

Equivalent form of $\xrightarrow{*}_S$

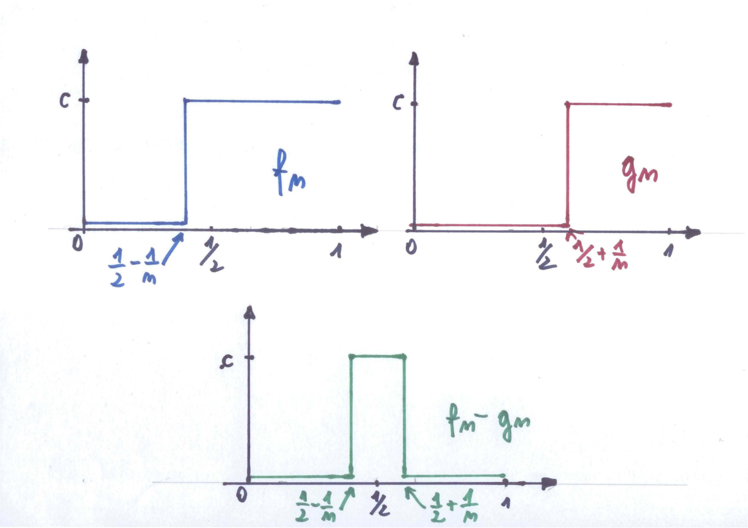
Theorem

$x_n \xrightarrow{*}_S x_0$ if, and only if, $x_n(1) \to x_0(1)$ and for each relatively τ-compact set $\mathcal{A} \subset \mathbb{A}$

$$\sup_{A \in \mathcal{A}} \Big| \int_0^1 (x_n(u) - x_0(u))\, dA(u) \Big| \to 0.$$

- Let σ be the (locally convex) topology on $\mathbb{D}$ given by the seminorm $\rho_1(x) = |x(1)|$ and the seminorms
$$\rho_{\mathcal{A}}(x) = \sup_{A \in \mathcal{A}} \Big| \int_0^1 x(u)\, dA(u) \Big|,$$
where $\mathcal{A}$ runs over relatively τ-compact subsets of $\mathbb{A}$.
- Then $x_n \xrightarrow{*}_S x_0$ if, and only if, $x_n \longrightarrow_\sigma x_0$.
- Corollary: $S \supset \sigma$.
- Conjecture: $S \equiv \sigma$. In other words, $(\mathbb{D}, S)$ is a linear topological space (in fact: locally convex LTS).

Addition is not continuous in J_1, but is sequentially continuous in S



The S topology and the J_1 topology

- $\mathbb{D}$ with the norm $\|\cdot\|_\infty$ is a Banach space, but non-separable.
- The J_1 topology of Skorokhod is metric separable and $(\mathbb{D}, J_1)$ is topologically complete, but $(\mathbb{D}, J_1)$ is not a linear topological space.
- Addition is not sequentially J_1-continuous!
- A discontinuous function cannot be approximated by continuous functions in the J_1 topology.
- OBSERVATION: the S topology is weaker than J_1.
- CONJECTURE: S is the finest linear topology on $\mathbb{D}$ "below" J_1.